

Dirichlet forms and semilinear elliptic equations with measure data

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Abstract

We propose a probabilistic definition of solutions of semilinear elliptic equations with (possibly nonlocal) operators associated with regular Dirichlet forms and with measure data. Using the theory of backward stochastic differential equations we prove the existence and uniqueness of solutions in the case where the right-hand side of the equation is monotone and satisfies mild integrability assumption, and the measure is smooth. We also study regularity of solutions under the assumption that the measure is smooth and has finite total variation. Some applications of our general results are given.

Keywords: Semilinear elliptic equation, Measure data, Dirichlet form, Backward stochastic differential equation.

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1 Introduction

Let E be a locally compact separable metric space, m be a Borel measure on E such that $\text{supp}[m] = E$, and let $(\mathcal{E}, D[\mathcal{E}])$ be a regular Dirichlet form on $L^2(E; m)$. Let A denote the operator corresponding to $(\mathcal{E}, D[\mathcal{E}])$, i.e. A is a nonpositive self-adjoint operator on $L^2(E; m)$ such that

$$D(A) \subset D[\mathcal{E}], \quad \mathcal{E}(u, v) = (-Au, v), \quad u \in D(A), v \in D[\mathcal{E}]$$

(see [10]). In the present paper we investigate semilinear elliptic equations of the form

$$-Au = f(\cdot, u) + \mu, \tag{1.1}$$

where $f : E \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function and μ is a smooth (possibly non-Radon) measure on E . Equations of the form (1.1) include semilinear equations for local operators (the model example is the Laplace operator subject to the Dirichlet or Neumann boundary conditions) as well as for nonlocal operators (the model example is the fractional Laplacian).

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There are many papers devoted to equations of the form (1.1) in case A is an elliptic second-order operator in divergence form and μ is a Radon measure (see, e.g., [2, 3, 8, 18] and the references given there). One of the main problems one encounters when considering such equations is to give proper definition of solutions which ensures uniqueness. To tackle this problem the so-called renormalized solutions (see [2, 8, 18]) and entropy solutions (see [3]) were introduced. Roughly speaking, these solutions are measurable functions whose truncates belong to the energy space, which satisfy an estimate on the decay of their energy on sets where they are large and satisfy (1.1) in the distributional sense for some wide class of test functions.

Our approach to (1.1) is quite different. In the paper we consider generalized probabilistic solutions of the problem (1.1). Let S denote the class of all smooth measures on E (see Section 4 for the definition; in particular every soft measure (see [9]) belongs to S). We first prove that if $\mu \in S$ and f is continuous and monotone with respect to the second variable and satisfies some mild integrability assumptions then the probabilistic solution of (1.1) exists and is unique in some class of functions having weak regularity properties. Then we show that if μ belongs to the class $\mathcal{M}_{0,b}$ of smooth measures of finite total variation and the form $(\mathcal{E}, D[\mathcal{E}])$ is transient then the solution has additional regularity properties.

To be more specific, let us denote by $\mathbb{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X, P_x)$ a Hunt process with life-time ζ associated with the form $(\mathcal{E}, D[\mathcal{E}])$ and let A^μ denote the continuous additive functional of \mathbb{X} which is in the Revuz correspondence with $\mu \in S$ (see [10]). By a probabilistic solution of (1.1) we mean a quasi-continuous function $u : E \rightarrow \mathbb{R}$ such that

$$u(x) = E_x \int_0^\zeta f(X_t, u(X_t)) dt + E_x \int_0^\zeta dA_t^\mu \quad (1.2)$$

for q.e. $x \in E$, i.e. u satisfies the nonlinear Feynman-Kac formula naturally associated with $(\mathcal{E}, D[\mathcal{E}])$ and μ, f . In the main theorem we prove that if $\mu \in S$, f satisfy the assumptions

- (A1) $f : E \times \mathbb{R} \rightarrow \mathbb{R}$ is measurable and $y \mapsto f(x, y)$ is continuous for every $x \in E$,
- (A2) $(f(x, y_1) - f(x, y_2))(y_1 - y_2) \leq 0$ for every $y_1, y_2 \in \mathbb{R}$ and $x \in E$,
- (A3') for every $r > 0$ the function $F_r(x) = \sup_{|y| \leq r} |f(x, y)|$, $x \in E$, is quasi- L^1 with respect to $(\mathcal{E}, D[\mathcal{E}])$, i.e. $t \mapsto F_r(X_t)$ belongs to $L_{loc}^1(\mathbb{R}^+)$ P_x -a.s. for q.e. $x \in E$,
- (A4') $E_x \int_0^\zeta |f(X_t, 0)| dt < \infty$, $E_x \int_0^\zeta d|A_t^\mu| < \infty$ for m -a.e. $x \in E$,

then there exists a unique solution of (1.2) in the class of quasi-continuous functions $u : E \rightarrow \mathbb{R}$ such that the process $t \mapsto u(X_t)$ is of Doob's class (D) under the measure P_x for q.e. $x \in E$. Moreover, for every $q \in (0, 1)$, $E_x \sup_{t \geq 0} |u(X_t)|^q < \infty$ for q.e. $x \in E$. We also show that (A3') is implied by (A3) and if $(\mathcal{E}, D[\mathcal{E}])$ is transient then (A4') is implied by (A4), where

$$(A3) \quad F_r \in L^1(E; m),$$

$$(A4) \quad f(\cdot, 0) \in L^1(E; m), \mu \in \mathcal{M}_{0,b}.$$

Let us remark that (A3'), (A4') are the minimal conditions which make it possible to define solutions of (1.1) by (1.2). Conditions (A1)–(A4) are widely used in L^1 -theory of nonlinear elliptic equations (see, e.g., [2]).

We have already mentioned that transiency of $(\mathcal{E}, D[\mathcal{E}])$ and additional assumptions on μ imply better regularity properties of the solution of (1.2). Namely, for transient forms, if u is a solution of (1.2) with μ, f satisfying (A4) then $f_u \in L^1(E; m)$ and

$$\|f_u\|_{L^1(E; m)} \leq \|f(\cdot, 0)\|_{L^1(E; m)} + \|\mu\|_{TV},$$

where $\|\mu\|_{TV}$ is the total variation norm of μ . Moreover, for every $k > 0$ the truncation of u defined by $T_k(u) = \min\{k, \max\{-k, u\}\}$ belongs to the extended Dirichlet space \mathcal{F}_e of $(\mathcal{E}, D[\mathcal{E}])$ and

$$\mathcal{E}(T_k(u), T_k(u)) \leq k(\|f_u\|_{L^1(E; m)} + \|\mu\|_{TV})$$

as well as

$$\mathcal{E}(\Phi_k(u), \Phi_k(u)) \leq \int_{\{|u| \geq k\}} |f_u(x)| m(dx) + \int_{\{|u| \geq k\}} d\mu,$$

where $\Phi_k(u) = T_1(u - T_k(u))$. These estimates are analogues of energy estimates for renormalized solutions. Up to now they were known for some classes of local operators (see, e.g., [2]). In general, u is even not locally integrable. We show that nevertheless $u \in L^1(E; m)$ in many interesting situations.

Another remarkable feature of probabilistic solutions in the transient case is that for $\mu \in \mathcal{M}_{0,b}$ they can be defined in purely analytic way, which resembles Stampacchia's way to defining solutions. Let $S_0^{(0)}$ denote the set of nonnegative Radon measures on E of finite 0-order energy integral and let $S_{00}^{(0)}$ be the subset of $S_0^{(0)}$ consisting of finite measures μ such that $\|U\mu\|_\infty < \infty$, where $U\mu$ is the (0-order) potential of μ (see [10]). We show that if $\mu \in \mathcal{M}_{0,b}$, u is quasi-continuous and $f(\cdot, u) \in L^1(E; m)$ then u is a probabilistic solution of (1.1) if and only if u is a solution of (1.1) in the sense of duality, i.e. $|\langle \nu, u \rangle| = |\int_E u d\nu| < \infty$ for every $\nu \in S_{00}^{(0)}$ and

$$\langle \nu, u \rangle = (f(\cdot, u), U\nu)_{L^2(E; m)} + \langle \mu, U\nu \rangle, \quad \nu \in S_{00}^{(0)},$$

If, in addition, $\mu \in S_0^{(0)}$ and $f(\cdot, u) \in L^2(E; m)$, then u is a weak solution of (1.1) in the sense that u belongs to the extended Dirichlet space \mathcal{F}_e and

$$\mathcal{E}(u, v) = (f(\cdot, u), v)_{L^2(E; m)} + \langle v, \mu \rangle, \quad v \in \mathcal{F}_e.$$

To apply our general results to concrete operator, one have to check that the form corresponding to it is a regular Dirichlet form and, to get better regularity of solutions, that the form is transient. In the paper we recall two classical examples of local and nonlocal operators associated with such forms, namely divergence form operators and Lévy diffusion generators. In the latter case our results lead to theorems on existence, uniqueness and regularity of equations of the form

$$-\psi(\nabla)u = f(\cdot, u) + \mu, \quad u|_{D^c} = 0,$$

where D is an open subset of \mathbb{R}^d and ψ is the Lévy-Khintchine symbol of some symmetric convolution semigroup of measures on \mathbb{R}^d . These theorems are new in the theory of semilinear equations with measure data. Note, however, that linear equations with fractional Laplacian and bounded smooth measure on the right-hand side are considered in [14]. The first example is provided mainly to illustrate that our approach allows one to treat in a unified way many interesting operators. It should be stressed, however, that even in the case of divergence form operators our results are new, because probabilistic approach enables us treat equations with measures which are not necessarily Radon measures. To our knowledge, our results for equations with Radon measures and possibly degenerating operator are also new. Some other possible applications of the main results of the paper are briefly indicated in Section 6.

Our proof of the main result on existence and uniqueness of solutions of (1.2) is probabilistic in nature. The idea is as follows. First we show that there exists a progressively measurable process Y of class (D) and a martingale M such that $Y_{T \wedge \zeta} \rightarrow 0$ as $T \rightarrow +\infty$ and for every $T > 0$,

$$Y_t = Y_{T \wedge \zeta} + \int_{t \wedge \zeta}^{T \wedge \zeta} f(X_s, Y_s) ds + \int_{t \wedge \zeta}^{T \wedge \zeta} dA_s^\mu - \int_{t \wedge \zeta}^{T \wedge \zeta} dM_s, \quad t \in [0, T]. \quad (1.3)$$

Then we set

$$u(x) = E_x Y_0, \quad x \in E \quad (1.4)$$

and show that u is quasi-continuous. Finally, using the Markov property we show that $u(X_t) = Y_t$, $t \geq 0$, P_x -a.s. for q.e. $x \in E$, which leads to (1.2). Let us point out that (1.4) means that the solution u of (1.2) is given by the first component of the solution (Y, M) of the backward stochastic differential equation (1.3). This representation is useful. For instance, it allows one to prove easily the comparison theorem for solutions of (1.1) and show that the solutions have some integrability properties.

The rest of the paper is organized as follows. In Sections 2 and 3 we prove theorems on existence, uniqueness and comparison of L^p -solutions of some general (non-Markovian) backward stochastic differential equations (BSDEs). In Section 4 we prove our main result on existence and uniqueness of probabilistic solutions of (1.1) in case $\mu \in S$. In Section 5 we investigate regularity of probabilistic solutions of (3.2) under the additional assumptions that \mathcal{E} is transient and $\mu \in \mathcal{M}_{0,b}$. In Section 6 some applications of general theorems proved in Sections 4 and 5 are given.

2 Generalized BSDEs with with constant terminal time

We assume as given a complete probability space (Ω, \mathcal{F}, P) equipped with a complete right continuous filtration $\{\mathcal{F}_t, t \geq 0\}$.

\mathcal{S} (resp. \mathcal{D}) is the space of all progressively measurable continuous (resp. càdlàg) processes. \mathcal{S}^p (resp. \mathcal{D}^p), $p > 0$, is the space of all processes $X \in \mathcal{S}$ (resp. $X \in \mathcal{D}$) such that

$$E \sup_{t \geq 0} |X_t|^p < \infty.$$

\mathcal{M} (resp. \mathcal{M}_{loc}) is the space of all càdlàg martingales (resp. càdlàg local martingales) and \mathcal{M}^p , $p > 0$, is the subspace of \mathcal{M} consisting of all martingales such that $E([M]_\infty)^{p/2} < \infty$.

\mathcal{V} is the space of all càdlàg progressively measurable processes of finite variation such that $V_0 = 0$. If $V \in \mathcal{V}$ then by $|V|_t$ we denote the variation of V on $[0, t]$ and by dV the random measure generated by the trajectories of V .

By \mathcal{T} we denote the set of all finite stopping times and by \mathcal{T}_t the set of all stopping times with values in $[0, t]$. We recall that a càdlàg adapted process Y is said to be of class (D) if the collection $\{Y_\tau, \tau \in \mathcal{T}\}$ is uniformly integrable. For a process Y of class (D) we set

$$\|Y\|_1 = \sup\{E|Y_\tau|, \tau \in \mathcal{T}\}.$$

For a process $X \in \mathcal{D}$ we set $X_{t-} = \lim_{s \nearrow t} X_s$ and $\Delta X_t = X_t - X_{t-}$ with the convention that $X_{0-} = 0$. Let $\{X^n\} \subset \mathcal{D}$, $X \in \mathcal{D}$. We say that $X^n \rightarrow X$ in ucp (uniformly on compacts in probability) if $\sup_{t \in [0, T]} |X_t^n - X_t| \rightarrow 0$ in probability P .

In the whole paper all equalities and inequalities and other relations between random elements are understood to hold P -a.s.. To avoid ambiguity we stress that writing $X_t = Y_t$, $t \in [0, T]$ we mean that $X_t = Y_t$, $t \in [0, T]$, P -a.s. whereas writing $X_t = Y_t$ for a.e. (resp. for every) $t \in [0, T]$ we mean that $X_t = Y_t$, P -a.s. for a.e. (resp. for every) $t \in [0, T]$. We also adopt the convention that $\int_a^b = \int_{(a, b]}$.

$T_k(x) = \min\{k, \max\{-k, x\}\}$, $x \in \mathbb{R}$. $x^+ = \max\{x, 0\}$, $x^- = \max\{-x, 0\}$ and

$$\hat{x} = \text{s\grave{g}n}(x), \quad \text{s\grave{g}n}(x) = \mathbf{1}_{x \neq 0} \frac{x}{|x|}, \quad x \in \mathbb{R}^d.$$

Definition. Let $\xi \in \mathcal{F}_T$, $V \in \mathcal{V}$ and let $f : [0, T] \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ be a function such that $f(\cdot, y)$ is progressively measurable for every $y \in \mathbb{R}$. We say that a pair (Y, M) is a solution of BSDE($\xi, f + dV$) on $[0, T]$ if $Y \in \mathcal{D}$, $M \in \mathcal{M}_{loc}$, $t \mapsto f(t, Y_t) \in L^1(0, T)$ and

$$Y_t = \xi + \int_t^T f(s, Y_s) ds + \int_t^T dV_s - \int_t^T dM_s, \quad t \in [0, T]. \quad (2.1)$$

We will need the following hypotheses.

- (H1) For every $t \in [0, T]$ the mapping $\mathbb{R} \ni y \mapsto f(t, y)$ is continuous.
- (H2) $(f(t, y) - f(t, y'))(y - y') \leq 0$ for every $t \geq 0$, $y, y' \in \mathbb{R}$.
- (H3) For every $r > 0$ the mapping $[0, T] \ni t \mapsto \sup_{|y| \leq r} |f(t, y) - f(t, 0)|$ belongs to $L^1(0, T)$.
- (H4) $E|\xi|^p + E(\int_0^T |f(t, 0)| dt)^p + E(\int_0^T d|V|_t)^p < \infty$.
- (A) There exists a nonnegative progressively measurable process $\{f_t\}$ such that

$$\forall (t, y) \in [0, T] \times \mathbb{R}, \quad \hat{y}f(t, y) \leq f_t.$$

Uniqueness of solutions of (2.1) follows from the following comparison result.

Proposition 2.1. *Let $(Y^1, M^1), (Y^2, M^2)$ be solutions of BSDE($\xi^1, f^1 + dV^1$) and BSDE($\xi^2, f^2 + dV^2$), respectively, such that Y^1, Y^2 are of class (D). Assume that $\xi^1 \leq \xi^2$, $dV^1 \leq dV^2$ and that*

$$f^2 \text{ satisfies (H2) and } f^1(t, Y_t^1) \leq f^2(t, Y_t^1) \text{ for a.e. } t \in [0, T] \quad (2.2)$$

or

$$f^1 \text{ satisfies (H2) and } f^1(t, Y_t^2) \leq f^2(t, Y_t^2) \text{ for a.e. } t \in [0, T]. \quad (2.3)$$

Then

$$Y_t^1 \leq Y_t^2, \quad t \in [0, T].$$

Proof. We give the proof in case (2.2) is satisfied. In case (2.3) is satisfied the proof is analogous and hence left to the reader. Let $\tau \in \mathcal{T}_T$. By the Itô-Tanaka formula,

$$\begin{aligned} (Y_{t \wedge \tau}^1 - Y_{t \wedge \tau}^2)^+ &\leq (Y_\tau^1 - Y_\tau^2)^+ + \int_{t \wedge \tau}^\tau \mathbf{1}_{\{Y_{s-}^1 > Y_{s-}^2\}} (f^1(s, Y_s^1) - f^2(s, Y_s^2)) ds \\ &\quad + \int_{t \wedge \tau}^\tau \mathbf{1}_{\{Y_{s-}^1 > Y_{s-}^2\}} d(V_s^1 - V_s^2) - \int_{t \wedge \tau}^\tau \mathbf{1}_{\{Y_{s-}^1 > Y_{s-}^2\}} d(M_s^1 - M_s^2). \end{aligned}$$

From the above and the assumptions,

$$(Y_{t \wedge \tau}^1 - Y_{t \wedge \tau}^2)^+ \leq (Y_\tau^1 - Y_\tau^2)^+ - \int_{t \wedge \tau}^\tau \mathbf{1}_{\{Y_{s-}^1 > Y_{s-}^2\}} d(M_s^1 - M_s^2), \quad t \in [0, T].$$

Let $\{\tau_k\}$ be a fundamental sequence for the local martingale $M^1 - M^2$. Since Y^1, Y^2 are of class (D), taking expectation of both sides of the above inequality with τ replaced by τ_k and then letting $k \rightarrow \infty$ shows that $E(Y_t^1 - Y_t^2)^+ \leq 0, t \in [0, T]$. This proves the proposition since Y^1, Y^2 are càdlàg processes. \square

Corollary 2.2. *Assume (H2). Then there exists at most one solution (Y, M) of BSDE($\xi, f + dV$) such that Y is of class (D).*

The following a priori estimates will be needed in the proof of existence of solutions of (2.1).

Lemma 2.3. *Let $p > 0$ and let (Y, M) be a solution of BSDE($\xi, f + dV$) such that $(Y, M) \in \mathcal{D}^p \otimes \mathcal{M}^p$ if $p \neq 1$ and Y is of class (D), $M \in \mathcal{M}_{loc}$ if $p = 1$. Then if (H2), (H4) are satisfied then*

$$E\left(\int_0^T |f(t, Y_t)| dt\right)^p \leq c_p E\left(|\xi|^p + \left(\int_0^T |f(t, 0)| dt\right)^p + \left(\int_0^T d|V|_t\right)^p + \mathbf{1}_{\{p \neq 1\}} [M]_T^{p/2}\right).$$

Proof. Let $\tau \in \mathcal{T}_T$. By the Itô-Tanaka formula,

$$-\int_0^\tau \text{sgn}(Y_t) f(t, Y_t) dt \leq |Y_\tau| - |Y_0| - \int_0^\tau \text{sgn}(Y_{t-}) dM_t + \int_0^\tau d|V|_t. \quad (2.4)$$

By (H3),

$$0 \leq -\int_0^\tau \text{sgn}(Y_t) (f(t, Y_t) - f(t, 0)) dt.$$

Combining this with (2.4) we get

$$\int_0^\tau |f(t, Y_t)| dt \leq \int_0^\tau |f(t, 0)| dt + |Y_\tau| - \int_0^\tau \text{sgn}(Y_{t-}) dM_t + \int_0^\tau d|V|_t,$$

from which one can easily deduce the desired inequality. \square

Remark 2.4. In case $p \neq 1$ the statement of Lemma 2.3 remains valid if we replace the condition $Y \in \mathcal{D}^p$ by the condition that $|Y|^p$ is of class (D).

Lemma 2.5. Let $p > 0$ and let (Y, M) be a solution of BSDE($\xi, f + dV$). Assume that (A) is satisfied and that

$$E\left(\int_0^T f_t dt\right)^p + E\left(\int_0^T d|V|_t\right)^p < \infty, \quad E \sup_{0 \leq t \leq T} |Y_t|^p < \infty.$$

If $p \in (0, 2]$ or $p > 2$ and M is locally in \mathcal{M}^p , then

$$E[M]_T^{p/2} \leq c_p E \left(\sup_{0 \leq t \leq T} |Y_t|^p + \left(\int_0^T f_t dt\right)^p + \left(\int_0^T d|V|_t\right)^p \right).$$

Proof. Let $\tau \in \mathcal{T}_T$. By Itô's formula,

$$|Y_0|^2 + [M]_\tau = |Y_\tau|^2 + 2 \int_0^\tau Y_t f(t, Y_t) dt + \int_0^\tau Y_{t-} dV_t - 2 \int_0^\tau Y_{t-} dM_t. \quad (2.5)$$

By the above and (A),

$$[M]_\tau \leq \sup_{0 \leq t \leq T} |Y_t|^2 + 2 \sup_{0 \leq t \leq T} |Y_t| \int_0^T f_t dt + 2 \sup_{0 \leq t \leq T} |Y_t| \int_0^T d|V|_t - 2 \int_0^\tau Y_{t-} dM_t.$$

By Young's inequality,

$$[M]_\tau^{p/2} \leq b_p \left(\sup_{0 \leq t \leq T} |Y_t|^p + \left(\int_0^T f_t dt\right)^p + \left(\int_0^T d|V|_t\right)^p + \left| \int_0^\tau Y_{t-} dM_t \right|^{p/2} \right). \quad (2.6)$$

Suppose that $E[M]_\tau^{p/2} < \infty$ for some $\tau \in \mathcal{T}_T$. Then by the Burkholder-Davis-Gundy inequality, Itô's isometry and again Young's inequality,

$$\begin{aligned} b_p E \left| \int_0^\tau Y_{t-} dM_t \right|^{p/2} &\leq c_p E \left[\int_0^\tau Y_{t-} dM_t \right]_\tau^{p/4} = c_p E \left(\int_0^\tau Y_{t-}^2 d[M]_t \right)^{p/4} \\ &\leq c_p E \left(\sup_{0 \leq t \leq T} |Y_t|^{p/2} [M]_\tau^{p/4} \right) \leq \frac{c_p^2}{2} E \sup_{0 \leq t \leq T} |Y_t|^p + \frac{1}{2} E [M]_\tau^{p/2}. \end{aligned}$$

Combining this with (2.6) gives

$$E[M]_\tau^{p/2} \leq d_p E \left(\sup_{0 \leq t \leq T} |Y_t|^p + \left(\int_0^T f_t dt\right)^p + \left(\int_0^T d|V|_t\right)^p \right). \quad (2.7)$$

To complete the proof it is enough to show that for every $p > 0$ there exists a stationary sequence $\{\tau_k\} \subset \mathcal{T}_T$ such that $M^{\tau_k} \in \mathcal{M}^p$, because then (2.7) holds true with τ replaced by τ_k , so letting $k \rightarrow \infty$ and using Fatou's lemma we obtain the required inequality. If $p > 2$ then the existence of $\{\tau_k\}$ follows from the assumption on M . If $p \in (0, 2]$ then any fundamental sequence for the local martingale $\int_0^\cdot Y_{t-} dM_t$ has the desired property. Indeed, if $\{\tau_k\}$ is such a sequence then by (2.5),

$$E[M]_{\tau_k}^{p/2} \leq c E \left(\sup_{0 \leq t \leq T} |Y_t|^p + \left(\int_0^T |f(t, 0)| dt\right)^p + \left| \int_0^{\tau_k} Y_{t-} dM_t \right| \right)$$

and the right-hand side of the above inequality is finite by the assumptions of the lemma and the very definition of the fundamental sequence. \square

Lemma 2.6. *Assume that (H1)–(H3) are satisfied and there exists $C > 0$ such that $\sup_{0 \leq t \leq T} |f(t, 0)| + |V|_T + |\xi| \leq C$. Then there exists a unique solution $(Y, M) \in \mathcal{D}^2 \otimes \mathcal{M}^2$ of BSDE($\xi, f + dV$).*

Proof. We first assume additionally that there is $L > 0$ such

$$|f(t, y) - f(t, y')| \leq L|y - y'| \quad (2.8)$$

for $t \in [0, T]$, $y, y' \in \mathbb{R}$. For $U \in \mathcal{D}^2$ let Y^U, M^U denote càdlàg versions of the processes \tilde{Y}^U, \tilde{M}^U defined by

$$\tilde{Y}_t^U = E(\xi + \int_0^T f(s, U_s) + \int_0^T dV_s | \mathcal{F}_t) - \int_0^t f(s, U_s) ds - \int_0^t dV_s$$

and

$$\tilde{M}_t^U = E(\xi + \int_0^T f(s, U_s) + \int_0^T dV_s | \mathcal{F}_t) - \tilde{Y}_0^U.$$

Then (Y^U, M^U) is a unique solution, in the class $\mathcal{D}^2 \otimes \mathcal{M}^2$, of the BSDE

$$Y_t^U = \xi + \int_t^T f(s, U_s) ds + \int_t^T dV_s - \int_t^T dM_s^U, \quad t \in [0, T]. \quad (2.9)$$

Therefore we may define the mapping $\Phi : \mathcal{D}^2 \otimes \mathcal{M}^2 \rightarrow \mathcal{D}^2 \otimes \mathcal{M}^2$ by putting

$$\Phi(U, N) = (Y^U, M^U).$$

By standard arguments (see, e.g., the proof of [21, Proposition 2.4]) one can show that Φ is contractive on the Banach space $(\mathcal{D}^2 \otimes \mathcal{M}^2, \|\cdot\|_\lambda)$, where

$$\|(Y, M)\|_\lambda = E \sup_{0 \leq t \leq T} e^{\lambda t} |Y_t|^2 + E \left[\int_0^\cdot e^{\lambda t} dM_t \right]_T$$

with suitably chosen $\lambda > 0$. Consequently, Φ has a fixed point $(Y, M) \in \mathcal{D}^2 \otimes \mathcal{M}^2$. Obviously (Y, M) is a unique solution of BSDE($\xi, f + V$). We now show how to dispense with the assumption (2.8). For $n \in \mathbb{N}$ put

$$f_n(t, y) = \inf_{x \in \mathbb{Q}} \{n|y - x| + f(t, x)\}.$$

It is an elementary check that

- (a) $|f_n(t, 0)| \leq C$, $|f_n(t, y) - f_n(t, y')| \leq n|y - y'|$ for all $t \in [0, T]$, $y, y' \in \mathbb{R}$,
- (b) $f_1(t, y) \leq f_n(t, y) \leq f(t, y)$ for all $t \in [0, T]$, $y, y' \in \mathbb{R}$ and $f_n(t, \cdot) \nearrow f(t, \cdot)$ uniformly on compact subsets of \mathbb{R} ,
- (c) $\sup_{|y| \leq r} |f_n(t, y)| \leq r + C + \sup_{|y| \leq r} |f(t, y)|$ for every $r > 0$.

By what has already been proved, for each $n \in \mathbb{N}$ there exists a unique solution $(Y^n, M^n) \in \mathcal{D}^2 \otimes \mathcal{M}^2$ of BSDE($\xi, f_n + dV$). By the Itô-Tanaka formula and (H2),

$$\begin{aligned} |Y_t^n| &\leq |\xi| + \int_t^T \text{sgn}(Y_s^n) f_n(s, Y_s^n) ds + \int_t^T \text{sgn}(Y_{s-}) dV_s - \int_t^T \text{sgn}(Y_{s-}^n) dM_s \\ &\leq |\xi| + \int_0^T |f_n(s, 0)| ds + \int_0^T d|V|_s - \int_t^T \text{sgn}(Y_{s-}^n) dM_s. \end{aligned}$$

By the above and the assumptions on ξ, f, V ,

$$|Y_t^n| \leq E(|\xi| + \int_0^T |f(s, 0)| ds + \int_0^T d|V|_s | \mathcal{F}_t) \leq C. \quad (2.10)$$

Moreover, by Lemma 2.1, $Y_t^n \leq Y_t^{n+1}$, $t \in [0, T]$. Therefore defining $Y_t = \sup_{n \geq 1} Y_t^n$, $t \in [0, T]$, we see that

$$Y_t^n \nearrow Y_t, \quad t \in [0, T], \quad E \int_0^T |Y_t^n - Y_t|^p dt \rightarrow 0, \quad p \geq 0. \quad (2.11)$$

By (2.10), (2.11), (H3) and (b), (c), $\int_0^T |f_n(t, Y_t^n) - f(t, Y_t)| dt \rightarrow 0$, while by Lemmas 2.3 and 2.5, $\sup_{n \geq 1} E(\int_0^T |f_n(t, Y_t^n)| dt)^2 < \infty$. Hence

$$E(\int_0^T |f_n(t, Y_t^n) - f(t, Y_t)| dt)^p \rightarrow 0 \quad (2.12)$$

for every $p \in (1, 2)$. Next, by Doob's L^p -inequality,

$$\begin{aligned} E \sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^p &\leq E \sup_{0 \leq t \leq T} \left(E(\int_0^T |f_n(s, Y_s^n) - f_m(s, Y_s^m)| ds | \mathcal{F}_t) \right)^p \\ &\leq c(p) E(\int_0^T |f_n(s, Y_s^n) - f_m(s, Y_s^m)| ds)^p, \end{aligned} \quad (2.13)$$

which when combined with (2.10)–(2.12) shows that $Y \in \mathcal{D}^2$ and $Y^n \rightarrow Y$ in \mathcal{D}^p , $p \in (1, 2)$. Since

$$Y_t^n = E(\xi + \int_t^T f_n(s, Y_s^n) ds + \int_t^T dV_s | \mathcal{F}_t),$$

using the fact that $Y^n \rightarrow Y$ in \mathcal{D}^p , $p \in (1, 2)$ and (2.12) we conclude that

$$Y_t = E(\xi + \int_t^T f(s, Y_s) ds + \int_t^T dV_s | \mathcal{F}_t), \quad t \in [0, T]. \quad (2.14)$$

Therefore the pair (Y, M) , where

$$M_t = E(\xi + \int_0^T f(s, Y_s) ds + \int_0^T dV_s | \mathcal{F}_t) - Y_0, \quad t \in [0, T] \quad (2.15)$$

is a solution of BSDE($\xi, f + dV$). The desired integrability properties of (Y, M) follow immediately from Lemma 2.5. \square

Theorem 2.7. *Assume that (H2), (H3) and (H4) with $p = 1$ are satisfied. Then there exists a unique solution (Y, M) of BSDE($\xi, f + dV$) such that $(Y, M) \in \mathcal{D}^q \otimes \mathcal{M}^q$, $q \in (0, 1)$, M is uniformly integrable and Y is of class (D).*

Proof. Write

$$\xi^n = T_n(\xi), \quad f_n(t, y) = f(t, y) - f(t, 0) + T_n(f(t, 0)), \quad V_t^n = \int_0^t \mathbf{1}_{\{|V|_s \leq n\}} dV_s.$$

By Lemma 2.6, for each $n \in \mathbb{N}$ there exists a unique solution $(Y^n, M^n) \in \mathcal{D}^2 \otimes \mathcal{M}^2$ of BSDE($\xi^n, f_n + dV^n$). In particular,

$$Y_t^n = E(\xi^n + \int_t^T f_n(s, Y_s^n) ds + \int_t^T dV_s^n | \mathcal{F}_t), \quad t \in [0, T]. \quad (2.16)$$

Write $\delta Y = Y^m - Y^n$, $\delta M = M^m - M^n$, $\delta \xi = \xi^m - \xi^n$, $\delta V = V^m - V^n$ for $m \geq n$. By the Itô-Tanaka formula and (H2),

$$\begin{aligned} |\delta Y_t| &\leq |\delta \xi| + \int_t^T \text{s\grave{g}n}(\delta Y_{s-})(f_m(s, Y_s^m) - f_n(s, Y_s^n)) ds \\ &\quad + \int_t^T \text{s\grave{g}n}(\delta Y_{s-}) d\delta V_s + \int_t^T \text{s\grave{g}n}(\delta Y_{s-}) d\delta M_s \\ &\leq |\delta \xi| + \int_0^T |f_m(s, Y_s^n) - f_n(s, Y_s^n)| ds + \int_0^T d|\delta V|_s + \int_t^T \text{s\grave{g}n}(\delta Y_{s-}) d\delta M_s. \end{aligned}$$

Conditioning both sides of the above inequality with respect to \mathcal{F}_t and using the definitions of ξ^n, f^n, V^n we get

$$|\delta Y_t| \leq E(\Psi^n | \mathcal{F}_t),$$

where

$$\Psi^n = |\xi| \mathbf{1}_{\{|\xi| > n\}} + \int_0^T |f(t, 0)| \mathbf{1}_{\{|f(t, 0)| > n\}} dt + \int_0^T \mathbf{1}_{\{|V|_t > n\}} d|V|_t.$$

From the above one can deduce that

$$\|\delta Y\|_1 \leq E\Psi^n$$

and, using [6, Lemma 6.1] (see also [24, Proposition IV.4.7]), that

$$E \sup_{0 \leq t \leq T} |\delta Y_t|^q \leq \frac{1}{1-q} E(\Psi^n)^q$$

for every $q \in (0, 1)$. Therefore there exists $Y \in \mathcal{D}^q$, $q \in (0, 1)$, such that Y is of class (D) and $Y^n \rightarrow Y$ in the norm $\|\cdot\|_1$ and in \mathcal{D}^q for $q \in (0, 1)$. From the last convergence and (H3) we conclude that

$$\int_0^T |f_n(t, Y_t^n) - f(t, Y_t)| dt \rightarrow 0.$$

By Lemmas 2.3 and 2.5,

$$\sup_{n \geq 1} E \left(\int_0^T |f_n(t, Y_t^n)| dt \right)^q < \infty$$

for every $q \in (0, 1)$. Therefore applying once again [6, Lemma 6.1] and letting $n \rightarrow \infty$ in (2.16) we see that Y satisfies (2.14) and hence the pair (Y, M) , where M is defined by (2.15), is a solution of $\text{BSDE}(\xi, f + dV)$. The integrability properties of M follow from Lemma 2.3, Lemma 2.5 and (2.15). \square

3 Generalized BSDEs with random terminal time

In this section $\zeta \in \mathcal{T}$, $V \in \mathcal{V}$ and $f : \mathbb{R}_+ \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is a function such that $f(\cdot, y)$ is progressively measurable for every $y \in \mathbb{R}$.

Definition. We say that a pair (Y, M) is a solution of $\text{BSDE}(\zeta, f + dV)$ if

- (a) $Y \in \mathcal{D}$, $Y_{t \wedge \zeta} \rightarrow 0$ as $t \rightarrow \infty$ and $M \in \mathcal{M}_{loc}$,
- (b) For every $T > 0$, $t \mapsto f(t, Y_t) \in L^1(0, T)$ and

$$Y_t = Y_{T \wedge \zeta} + \int_{t \wedge \zeta}^{T \wedge \zeta} f(s, Y_s) ds + \int_{t \wedge \zeta}^{T \wedge \zeta} dV_s - \int_{t \wedge \zeta}^{T \wedge \zeta} dM_s, \quad t \in [0, T]. \quad (3.1)$$

Let us observe that from the above definition it follows that $Y_t = Y_{t \wedge \zeta}$ for every $t \geq 0$.

We first state the analogues of Proposition 2.1 and Corollary 2.2.

Proposition 3.1. *Let $(Y^1, M^1), (Y^2, M^2)$ be solutions of $\text{BSDE}(\zeta, f^1 + dV^1)$ and $\text{BSDE}(\zeta, f^2 + dV^2)$, respectively, such that Y^1, Y^2 are of class (D). If $dV^1 \leq dV^2$ and either (2.2) or (2.3) is satisfied then $Y_t^1 \leq Y_t^2$, $t \geq 0$.*

Proof. Assume that (2.2) is satisfied. Let $Y = Y^1 - Y^2$, $M = M^1 - M^2$ and let $\tau \in \mathcal{T}$. By the Itô-Tanaka formula and (H2), for every $T > 0$ we have

$$\begin{aligned} Y_t^+ &\leq Y_{T \wedge \tau \wedge \zeta}^+ + \int_{t \wedge \tau \wedge \zeta}^{T \wedge \tau \wedge \zeta} \mathbf{1}_{\{Y_{s-}^1 > Y_{s-}^2\}} (f(s, Y_s^1) - f(s, Y_s^2)) ds \\ &\quad + \int_{t \wedge \tau \wedge \zeta}^{T \wedge \tau \wedge \zeta} \mathbf{1}_{\{Y_{s-}^1 > Y_{s-}^2\}} dM_s \leq Y_{T \wedge \tau \wedge \zeta}^+ + \int_{t \wedge \tau \wedge \zeta}^{T \wedge \tau \wedge \zeta} \mathbf{1}_{\{Y_{s-}^1 > Y_{s-}^2\}} dM_s, \quad t \geq 0. \end{aligned}$$

Let $\{\tau_k\}$ be a fundamental sequence for M . Since Y is of class (D), taking expectation of both sides of the above inequality with τ replaced by τ_k and then letting $k \rightarrow \infty$ we see that $EY_t^+ \leq EY_{T \wedge \zeta}^+$, $t \geq 0$. Therefore letting $T \rightarrow \infty$ and using once again the fact that Y is of class (D) we conclude that $Y_t = 0$, $t \geq 0$. In case (2.3) is satisfied the proof is analogous. \square

Corollary 3.2. *Assume (H2). Then there exists at most one solution (Y, M) of $\text{BSDE}(\zeta, f + dV)$ such that Y is of class (D).*

Lemma 3.3. *Let (Y, M) be a solution of $\text{BSDE}(\xi, f + dV)$ on $[0, T]$. Assume additionally that ξ is \mathcal{F}_τ measurable for some $\tau \in \mathcal{T}_T$, $f(\cdot, y) = 0$ on the interval $(\tau, T]$ and $\int_\tau^T d|V|_t = 0$. Then $(Y_{t \wedge \tau}, M_{t \wedge \tau}) = (Y_t, M_t)$, $t \in [0, T]$.*

Proof. Let $\{\sigma_k\}$ be a fundamental sequence for M . By the assumptions, for every $k \in \mathbb{N}$ and $\delta \in \mathcal{T}_T$ such that $\delta \geq \tau$,

$$Y_\delta = Y_{\delta \vee \sigma_k} - \int_\delta^{\delta \vee \sigma_k} dM_s.$$

Since Y is of class (D) and ξ is \mathcal{F}_τ -measurable, it follows that

$$Y_\delta = E(Y_{\delta \vee \sigma_k} | \mathcal{F}_\delta) \rightarrow E(\xi | \mathcal{F}_\delta) = \xi. \quad (3.2)$$

By Itô's formula,

$$|Y_\tau|^2 + \int_\tau^{t \vee \tau} d[M]_s = |Y_{t \vee \tau}|^2 - 2 \int_\tau^{t \vee \tau} Y_{s-} dM_s.$$

By the above and (3.2),

$$\int_\tau^{t \vee \tau} d[M]_s = -2 \int_\tau^{t \vee \tau} Y_{s-} dM_s, \quad t \in [0, T],$$

which implies that $M_{t \wedge \tau} = M_\tau$, $t \in [0, T]$. Since $Y_t = \xi - \int_t^T dM_s$ for $t \in [\tau, T]$, we get the desired result. \square

We can now prove our main result on existence and uniqueness of solutions of (3.1).

Theorem 3.4. *Assume that $E \int_0^\zeta d|V|_t + E \int_0^\zeta |f(t, 0)| dt < \infty$ and that f satisfies (H1)–(H3) for every $T > 0$. Then there exists a unique solution (Y, M) of BSDE($\zeta, f + dV$) such that $(Y, M) \in \mathcal{D}^q \otimes \mathcal{M}^q$ for $q \in (0, 1)$, M is uniformly integrable martingale and Y is of class (D).*

Proof. By Theorem 2.7, for each $n \in \mathbb{N}$ there exists a unique solution (Y^n, M^n) of the BSDE

$$Y_t^n = \int_t^n \mathbf{1}_{[0, \zeta]}(s) f(s, Y_s^n) ds + \int_t^n \mathbf{1}_{[0, \zeta]}(s) dV_s - \int_t^n dM_s^n, \quad t \in [0, n] \quad (3.3)$$

such that $(Y^n, M^n) \in \mathcal{D}^q \otimes \mathcal{M}^q$, $q \in (0, 1)$, M^n is uniformly integrable martingale and Y^n is of class (D). Let us put $(Y_t^n, M_t^n) = (0, M_n^n)$ for $t \geq n$. Then by Lemma 3.3,

$$(Y_t^n, M_t^n) = (Y_{t \wedge \zeta}^n, M_{t \wedge \zeta}^n), \quad t \geq 0. \quad (3.4)$$

For $m \geq n$ put $\delta Y = Y^m - Y^n$, $\delta M = M^m - M^n$ and

$$\begin{aligned} \varphi(t) &= \int_0^t \mathbf{1}_{[0, \zeta \wedge m]}(s) f(s, Y_s^m) ds + \int_0^t \mathbf{1}_{[0, \zeta \wedge m]}(s) dV_s \\ &\quad - \int_0^t \mathbf{1}_{[0, \zeta \wedge n]}(s) f(s, Y_s^n) ds - \int_0^t \mathbf{1}_{[0, \zeta \wedge n]}(s) dV_s. \end{aligned}$$

Then by the Itô-Tanaka formula,

$$|\delta Y_t| \leq \int_t^m \text{s\grave{g}n}(\delta Y_{s-}) d\varphi(s) - \int_t^m \text{s\grave{g}n}(\delta Y_{s-}) d\delta M_s, \quad t \in [0, m].$$

Conditioning both sides of the above inequality with respect to \mathcal{F}_t we get

$$|\delta Y_t| \leq E\left(\int_t^m \text{sgn}(\delta Y_{s-}) d\varphi(s) | \mathcal{F}_t\right), \quad t \in [0, m]. \quad (3.5)$$

Since for every $t \in [n, m]$,

$$\int_t^m \text{sgn}(\delta Y_{s-}) d\varphi(s) = \int_{t \wedge \zeta}^{m \wedge \zeta} \text{sgn}(Y_{s-}^m) f(s, Y_s^m) ds + \int_{t \wedge \zeta}^{m \wedge \zeta} \text{sgn}(Y_{s-}^m) dV_s, \quad (3.6)$$

using [6, Lemma 6.1] and (H2) we deduce from (3.5) that for every $q \in (0, 1)$,

$$E \sup_{n \leq t \leq m} |\delta Y_t|^q \leq \frac{1}{1-q} E\left(\int_{n \wedge \zeta}^{\zeta} |f(s, 0)| ds + \int_{n \wedge \zeta}^{\zeta} d|V|_s\right)^q. \quad (3.7)$$

Observe that for $t \in [0, n]$,

$$\int_t^m \text{sgn}(\delta Y_{s-}) d\varphi(s) = \int_n^m \text{sgn}(\delta Y_{s-}) d\varphi(s) + \int_t^n \text{sgn}(\delta Y_{s-}) d\varphi(s) \quad (3.8)$$

and

$$\varphi(t) = \int_0^t \mathbf{1}_{[0, \zeta]}(s) (f(s, Y_s^m) - f(s, Y_s^n)) ds. \quad (3.9)$$

From (3.5), (3.8), (3.9) and [6, Lemma 6.1] it follows that for every $q \in (0, 1)$,

$$E \sup_{0 \leq t \leq n} |\delta Y_t|^q \leq \frac{1}{1-q} E\left(\int_{n \wedge \zeta}^{\zeta} |f(s, 0)| ds + \int_{n \wedge \zeta}^{\zeta} d|V|_s\right)^q. \quad (3.10)$$

Combining (3.7) with (3.10) we see that for every $q \in (0, 1)$,

$$E \sup_{t \geq 0} |\delta Y_t|^q \leq \frac{1}{1-q} E\left(\int_{n \wedge \zeta}^{\zeta} |f(s, 0)| ds + \int_{n \wedge \zeta}^{\zeta} d|V|_s\right)^q. \quad (3.11)$$

Using once again (3.6), (3.8), (3.9) we deduce from (3.5) that

$$\|\delta Y\|_1 \leq \frac{1}{1-q} E\left(\int_{n \wedge \zeta}^{\zeta} |f(s, 0)| ds + \int_{n \wedge \zeta}^{\zeta} d|V|_s\right)^q.$$

Therefore there exists $Y \in \mathcal{D}^q$, $q \in (0, 1)$, such that Y is of class (D), $Y^n \rightarrow Y$ in the norm $\|\cdot\|_1$ and in \mathcal{D}^q for $q \in (0, 1)$. By the latter convergence and (3.4), $Y_{t \wedge \zeta} \rightarrow 0$ as $t \rightarrow \infty$. By Lemma 2.5, for any $m \geq n \geq T > 0$,

$$E[\delta M]_T^{q/2} \leq c_q E \sup_{0 \leq t \leq T} |\delta Y_t|^q, \quad q \in (0, 1).$$

From this and (3.11) it follows that there exists $M \in \mathcal{M}$ such that for every $q \in (0, 1)$ and $T > 0$,

$$E[M^n - M]_T^{q/2} \rightarrow 0 \quad (3.12)$$

as $n \rightarrow \infty$. By (H1), (H3), (3.11) and the Lebesgue dominated convergence theorem,

$$\int_0^T |f(s, Y_s^n) - f(s, Y_s)| ds \rightarrow 0 \quad (3.13)$$

as $n \rightarrow \infty$. By the definition of the processes (Y^n, M^n) , for every $T > 0$,

$$Y_t^n = Y_{T \wedge \zeta}^n + \int_{t \wedge \zeta}^{T \wedge \zeta} f(s, Y_s^n) ds + \int_{t \wedge \zeta}^{T \wedge \zeta} dV_s - \int_{t \wedge \zeta}^{T \wedge \zeta} dM_s^n, \quad t \geq 0.$$

Therefore letting $n \rightarrow \infty$ and using (3.11)–(3.13) we see that (Y, M) satisfies (3.1). What is left is to show integrability properties of M . That $M \in \mathcal{M}^q$, $q \in (0, 1)$, follows from the fact that $Y \in \mathcal{D}^q$ for $q \in (0, 1)$ and Lemma 2.5. By Lemmas 2.3 and 2.5, $E \int_0^\zeta |f(s, Y_s)| ds < \infty$. Using this, the fact that Y is of class (D) and $Y_{t \wedge \zeta} \rightarrow 0$ as $t \rightarrow \infty$ it is easy to deduce from (3.1) that M has the form

$$M_t = E\left(\int_0^\zeta f(s, Y_s) ds + \int_0^\zeta dV_s \middle| \mathcal{F}_t\right) - Y_0, \quad t \geq 0.$$

Thus, M is closed and hence uniformly integrable. \square

Let E be a Radon space and let $\mathbb{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X, \theta_t, \zeta, P_x,)$ be a right process (with translation operators θ_t and life-time ζ) on E . Suppose we are given a measurable function $f : E \times \mathbb{R} \rightarrow \mathbb{R}$ and a finite variation additive functional V of \mathbb{X} . Then for $x \in E$, $r \geq 0$ we put $\zeta^r = \zeta + r$, $V^r = V_{-r}$ and we define $f^r : [r, +\infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ by putting $f^r(t, y)(\omega) = f(X_{t-r}(\omega), y)$.

Proposition 3.5. *Assume that for every $r \geq 0$ the function f^r satisfies (H2) and that there exists a solution $(Y^{r,x}, M^{r,x}) = (Y^r, M^r)$ of BSDE($\zeta^r, f^r + dV^r$) on $[r, \infty)$, defined on the space $(\Omega, \mathcal{F}, \mathcal{F}_{-r}, P_x)$, such that Y^r is of class (D). Then for every $h \geq 0$,*

- (i) $(M_t^{r+h} \circ \theta_h, \mathcal{F}_{t-r}, t \geq r+h)$ is a martingale under P_x ,
- (ii) $(Y_t^{r+h} \circ \theta_h, M_t^{r+h} \circ \theta_h) = (Y_t^r, M_t^r)$, $t \geq r+h$, P_x -a.s.,
- (iii) $(Y_{t+h}^{r+h}, M_{t+h}^{r+h}) = (Y_t^r, M_t^r)$, $t \geq r \geq 0$, $h \geq 0$, P_x -a.s..

Proof. (i) Let $N_t = M_{t+r+h}^{r+h}$. By the assumption, $(N_t, \mathcal{F}_t, t \geq 0)$ is a local martingale. Hence, by [26, Proposition 50.19], $(N_{t-h} \circ \theta_h \mathbf{1}_{[h, +\infty)}(t), \mathcal{F}_t, t \geq 0)$ is again a martingale. But $N_{t-h} \circ \theta_h = M_{t+r}^{r+h} \circ \theta_h$, $t \geq h$ which implies (i).

(ii) By the assumption,

$$Y_t^r = Y_T^r + \int_{t \wedge \zeta^r}^{T \wedge \zeta^r} f(X_{s-r}, Y_s^r) ds + \int_{t \wedge \zeta^r}^{T \wedge \zeta^r} dV_s^r - \int_{t \wedge \zeta^r}^{T \wedge \zeta^r} dM_s^r, \quad t \in [r, T]$$

and

$$\begin{aligned} Y_t^{r+h} &= Y_T^{r+h} + \int_{t \wedge \zeta^{r+h}}^{T \wedge \zeta^{r+h}} f(X_{s-r-h}, Y_s^{r+h}) ds \\ &\quad + \int_{t \wedge \zeta^{r+h}}^{T \wedge \zeta^{r+h}} dV_s^{r+h} - \int_{t \wedge \zeta^{r+h}}^{T \wedge \zeta^{r+h}} dM_s^{r+h}, \quad t \in [r+h, T]. \end{aligned}$$

Hence

$$\begin{aligned} Y_t^{r+h} \circ \theta_h &= Y_T^{r+h} \circ \theta_h + \int_{t \wedge \zeta^{r,h}}^{T \wedge \zeta^{r,h}} f(X_{s-r}, Y_s^{r+h} \circ \theta_h) ds \\ &\quad + \int_{t \wedge \zeta^{r,h}}^{T \wedge \zeta^{r,h}} d(V_s^{r+h} \circ \theta_h) - \int_{t \wedge \zeta^{r,h}}^{T \wedge \zeta^{r,h}} d(M_s^{r+h} \circ \theta_h), \quad t \in [r+h, T], \end{aligned}$$

where $\zeta^{r,h} = (\zeta - h)^+ + r + h$. By Proposition 3.5, $(M_t^{r+h} \circ \theta_h, \mathcal{F}_{t-r}, t \geq r+h)$ is a martingale. Since V is additive, $d(V_s^{r+h} \circ \theta_h) = dV_s^r$. Observe also that if $\zeta \geq h$ then $\zeta^{r,h} = \zeta^r$ and if $\zeta \leq h$ then $T \wedge \zeta^r = t \wedge \zeta^r, T \wedge \zeta^{h,r} = t \wedge \zeta^{h,r}$ for $t \in [r+h, T]$. Therefore,

$$\begin{aligned} Y_t^{r+h} \circ \theta_h &= Y_T^{r+h} \circ \theta_h + \int_{t \wedge \zeta^r}^{T \wedge \zeta^r} f(X_{s-r}, Y_s^{r+h} \circ \theta_h) ds \\ &\quad + \int_{t \wedge \zeta^r}^{T \wedge \zeta^r} dV_s^r - \int_{t \wedge \zeta^r}^{T \wedge \zeta^r} d(M_s^{r+h} \circ \theta_h), \quad t \in [r+h, T]. \end{aligned}$$

We see that $(Y^r, M^r), (Y^{r+h} \circ \theta_h, M^{r+h} \circ \theta_h)$ are solutions of BSDE($\zeta^r, f^r + dV^r$) on $[r+h, +\infty)$ defined on the space $(\Omega, \mathcal{F}, \mathcal{F}_{-r}, P_x)$. Therefore (ii) follows from Corollary 3.2. Since the proof of (iii) is analogous to that of (ii), we omit it. \square

Remark 3.6. Let B be a Borel subset of E and for $x \in B$ let the pair (Y^x, M^x) be a unique solution of BSDE($\xi, f + dV$) of Theorem 3.4 defined on the filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$. Then there exists a pair (Y, M) of (\mathcal{F}_t) adapted càdlàg processes such that $(Y_t, M_t) = (Y_t^x, M_t^x)$, $t \geq 0$, P_x -a.s. for every $x \in B$. This follows from the construction of solutions (Y^x, M^x) and repeated application of Lemmas A.3.3 and A.3.5 in [10]. Indeed, let $(Y^{x,n}, M^{x,n})$ be a solution of (3.3) on $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$. Since $Y^{x,n} \rightarrow Y^x$ in probability P_x for $x \in B$, to prove the desired result it suffices to show that there exists a pair (Y^n, M^n) of (\mathcal{F}_t) adapted càdlàg processes such that $(Y_t^n, M_t^n) = (Y_t^{x,n}, M_t^{x,n})$, $t \geq 0$, P_x -a.s. for every $x \in B$. But the solution of (3.3) is a limit in probability of solutions of equations considered in Lemma 2.6 (see the proof of Theorem 2.7), and solutions of equations considered in Lemma 2.6 are limits of Picard iterations of solutions of linear equations of the form (2.9). Using [10, Lemma A.3.5] one can find independent of x solutions of these linear equations. Consequently, using [10, Lemma A.3.3] we can find (Y^n, M^n) having the desired properties.

4 Probabilistic solutions of equations with measure data

In the rest of the paper we assume that

- E is a locally compact separable metric space and m is a positive Radon measure on E such that $\text{supp}[m] = E$,
- $(\mathcal{E}, D[\mathcal{E}])$ is a regular Dirichlet form on $L^2(E; m)$.

Let $D[\mathcal{E}]$ be a dense linear subspace of $L^2(E; m)$ and let \mathcal{E} be a nonnegative symmetric bilinear form on $D[\mathcal{E}] \times D[\mathcal{E}]$. Set $\mathcal{E}_\alpha(u, v) = \mathcal{E}(u, v) + \alpha(u, v)$ for $u, v \in D[\mathcal{E}]$, $\alpha > 0$.

Let us recall that $(\mathcal{E}, D[\mathcal{E}])$ is called a Dirichlet form if it is closed, i.e. $D[\mathcal{E}]$ is complete under the norm \mathcal{E}_1 , and Markovian, i.e. if $u \in D[\mathcal{E}]$ and v is a normal contraction of u then $v \in D[\mathcal{E}]$ and $\mathcal{E}(u, u) \leq \mathcal{E}(v, v)$ (a function v is called a normal contraction of u if $|v(x) - v(y)| \leq |u(x) - u(y)|$ and $|v(x)| \leq |u(x)|$ for $x, y \in E$).

A Dirichlet form $(\mathcal{E}, D[\mathcal{E}])$ is called regular if the space $D[\mathcal{E}] \cap C_0(E)$ is dense in $D[\mathcal{E}]$ with respect to the norm \mathcal{E}_1 and dense in $C_0(E)$ with respect to the uniform convergence topology, where $C_0(E)$ is the space of continuous functions on E with compact support.

Let $\text{cap} : 2^E \rightarrow \mathbb{R}^+$ denote the Choquet capacity associated with the form $(\mathcal{E}, D[\mathcal{E}])$ (see [10, Chapter 2]). In the sequel we say that a statement depending on $x \in E$ holds quasi-everywhere (“q.e.” for short) on E if there is a set $B \subset E$ of capacity zero such that the statement is true for every $x \in B$.

A function $u : E \rightarrow \mathbb{R}$ is called quasi-continuous if for every $\varepsilon > 0$ there exists an open set $U \subset E$ such that $\text{cap}(U) < \varepsilon$ and $u|_{E \setminus U}$ is continuous. It is known that each $u \in D[\mathcal{E}]$ admits a quasi-continuous m -version (see [10, Theorem 2.1.3]). In the sequel we always consider a quasi-continuous version of u if it has such a version.

Let $\mathbb{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X, \theta_t, P_x)$ be a (unique) Hunt process associated with $(\mathcal{E}, D[\mathcal{E}])$ (see [10, Chapter 7]). In what follows by ζ we denote the life-time of X , i.e. $\zeta = \inf\{t \geq 0; X_t = \Delta\}$, where Δ is the one-point compactification of E . If E is already compact, Δ is adjoined as an isolated point.

For $B \subset E$ we set

$$\sigma_B(\omega) = \inf\{t > 0; X_t(\omega) \in B\}.$$

A set $B \subset E$ is called nearly Borel if for each finite nonnegative Borel measure ν on E there exist Borel sets B_1, B_2 such that $B_1 \subset B \subset B_2$ and $P_\nu(\exists t \geq 0; X_t \in B_2 \setminus B_1) = 0$, where $P_\nu(\cdot) = \int P_x(\cdot) \nu(dx)$. A set $N \subset E$ is called exceptional if there exists a nearly Borel set \tilde{N} such that $N \subset \tilde{N}$ and $P_m(\sigma_{\tilde{N}} < +\infty) = 0$. By [10, Theorem 4.2.1], a set $N \subset E$ is exceptional iff $\text{cap}(N) = 0$.

Let $\mathcal{B}(E)$ ($\mathcal{B}^n(E)$) denote the space of all Borel (nearly Borel) measurable functions $u : E \rightarrow \mathbb{R}$ and let \mathcal{C} denote the space of all $u \in \mathcal{B}^n(E)$ for which there exists an exceptional Borel set $B \subset E$ such that the process $t \rightarrow u(X_t)$ is right continuous and $t \rightarrow u(X_{t-})$ is left continuous on $[0, \zeta)$ under P_x for every $x \in B^c$. By [10, Theorem 4.2.2] and [16, Theorem 5.29], $u \in \mathcal{B}^n(E)$ is quasi-continuous iff it belongs to \mathcal{C} .

An increasing sequence $\{F_n\}$ of closed subsets of E is called a generalized nest if

$$\lim_{n \rightarrow \infty} \text{cap}(K \setminus F_n) = 0 \tag{4.1}$$

for any compact set $K \subset E$. $\{F_n\}$ is called a nest if (4.1) holds with E in place of K .

Recall that a Borel measure μ on E is called smooth if its total variation $|\mu|$ charges no set of zero capacity and there exists a generalized nest $\{F_n\}$ such that $|\mu|(F_n) < \infty$ for every $n \in \mathbb{N}$. In the sequel, given a nonnegative Borel measure μ on E and $f \in \mathcal{B}^+(E)$ we put

$$\int_E f d\mu = \langle f, \mu \rangle.$$

By $f \cdot \mu$ we denote the Borel measure on E such that $\frac{df \cdot \mu}{d\mu} = f$.

Let S denote the set of all nonnegative smooth measures on E . It is known (see [10, Chapter 5]) that for every measure $\mu \in S$ there exists a unique positive continuous

additive functional (PCAF) A^μ of \mathbb{X} which is in the Revuz correspondence with μ , i.e. for every bounded nonnegative $f \in \mathcal{B}(E)$,

$$\lim_{t \searrow 0} \frac{1}{t} E_m \int_0^t f(X_s) dA_s^\mu = \int_E f(x) \mu(dx).$$

Lemma 4.1. *Let A be a PCAF of \mathbb{X} . Then for every stopping time τ , $E_x A_\tau$ is finite for m -a.e. $x \in E$ iff $E_x A_\tau$ is finite for q -e. $x \in E$.*

Proof. Sufficiency follows from the definition of the capacity. To prove necessity let us assume that $E_x A_\tau < +\infty$ for m -a.e. $x \in E$ and set $B = \{x \in E; E_x A_\tau < \infty\}$. Since \mathbb{X} is a Hunt process, B is a Borel set. Let K be a compact set such that $K \subset B$. Since \mathbb{X} is strong Markov and A is additive, for m -a.e. $x \in E$ we have

$$\begin{aligned} P_x(\sigma_K < \infty) &= P_x(E_{X_{\sigma_K}} A_\tau = \infty) = P_x(E_x(A_\tau \circ \theta_{\sigma_K} | \mathcal{F}_{\sigma_K}) = \infty) \\ &= P_x(E_x(A_\tau - A_{\sigma_K \wedge \tau} | \mathcal{F}_{\sigma_K}) = \infty) = 0. \end{aligned}$$

Thus, $P_m(\sigma_K < \infty) = 0$ or, equivalently, $\text{cap}(K) = 0$. Since this holds for arbitrary compact set $K \subset B$ and cap is a Choquet capacity, $\text{cap}(B) = 0$. \square

Lemma 4.2. *Assume that A is PCAF of \mathbb{X} such that $E_x A_\zeta < \infty$ for m -a.e. $x \in E$. Then the function*

$$u(x) = E_x A_\zeta, \quad x \in E$$

is quasi-continuous.

Proof. By Lemma 4.1, $u(x) < \infty$ q.e.. Hence, by [10, Theorem 4.1.1], without loss of generality we may assume that $B = \{x \in E; u(x) = \infty\}$ is properly exceptional. Since A is PCAF of \mathbb{X} , by [10, Theorem 5.1.4] there exists a unique measure $\mu \in \mathcal{S}$ such that $A = A^\mu$. Let $\{F_n\}$ be a generalized nest such that $\mathbf{1}_{F_n} \mu \in \mathcal{S}_{00}$ (see [10, Theorem 2.2.4]). Then for each $n \in \mathbb{N}$ the function u_n defined by

$$u_n(x) \equiv E_x \int_0^{+\infty} e^{-t/n} \mathbf{1}_{F_n}(X_t) dA_t = E_x \int_0^{+\infty} e^{-t/n} dA_t^{\mathbf{1}_{F_n} \mu} < \infty, \quad x \in E$$

is quasi-continuous (see Theorems 5.1.1 and 5.1.6 in [10]). Let us observe that

$$u_n(x) \nearrow u(x), \quad x \in B^c. \quad (4.2)$$

Indeed, since E is locally compact separable metric space, to prove (4.2) it suffices to show that for every compact set $K \subset E$,

$$\lim_{n \rightarrow +\infty} E_x \int_0^{+\infty} \mathbf{1}_{K \setminus F_n}(X_t) dA_t = 0. \quad (4.3)$$

By [10, Theorem 4.2.1], $p_{K \setminus F_n}^1(x) \rightarrow 0$, q.e., where $p_{K \setminus F_n}^1(x) = E_x e^{-\sigma_{K \setminus F_n}}$, $x \in E$. In view of [10, Theorem 4.1.1], without loss of generality we may assume that $p_{K \setminus F_n}^1(x) \rightarrow 0$ for every $x \in B^c$. The last convergence implies that for every $x \in B^c$,

$$\mathbf{1}_{K \setminus F_n}(X_t) \rightarrow 0, \quad t \geq 0, \quad P_x\text{-a.s.} \quad (4.4)$$

Using this, the definition of B and the Lebesgue dominated convergence theorem we get (4.3), and hence (4.2). From (4.4) it also follows that for every $x \in B^c$,

$$\lim_{n,m \rightarrow \infty} E_x \int_0^\infty \mathbf{1}_{F_n \Delta F_m}(X_t) dA_t = 0. \quad (4.5)$$

By the Markov property and [6, Lemma 6.1],

$$E_x \sup_{t \geq 0} |u_n(X_t) - u_m(X_t)|^q \leq \frac{1}{1-q} E_x \left(\int_0^\infty \mathbf{1}_{F_n \Delta F_m}(X_t) dA_t \right)^q, \quad q \in (0, 1).$$

Combining this with (4.2), the fact that B is properly exceptional set and (4.5) shows that for every $x \in B$,

$$\lim_{n \rightarrow \infty} E_x \sup_{t \geq 0} |u_n(X_t) - u(X_t)|^q = 0. \quad (4.6)$$

Since u_n is quasi-continuous, $u_n \in \mathcal{C}$. From this and (4.6) it may be concluded that $u \in \mathcal{C}$, i.e. u is quasi-continuous. \square

Let A denote the unique nonpositive self-adjoint operator on $L^2(E; m)$ such that

$$D(A) \subset D[\mathcal{E}], \quad \mathcal{E}(u, v) = (-Au, v), \quad u \in D(A), v \in D[\mathcal{E}]$$

(see [10, Corollary 1.3.1]) and let $\mathbb{X} = (\Omega, \mathcal{F}, \mathcal{F}_t, X, P_x)$ be a Hunt process with life-time ζ associated with $(\mathcal{E}, D[\mathcal{E}])$.

Definition. Let $\mu \in S$ be a measure such that $E_x |A^\mu|_\zeta < \infty$ for m -a.e. $x \in E$, where A^μ is the CAF of \mathbb{X} associated with μ . We say that a quasi-continuous function $u : E \rightarrow \mathbb{R}$ is a probabilistic solution of the equation

$$-Au = f_u + \mu, \quad (4.7)$$

where $f_u = f(\cdot, u)$, if $E_x \int_0^\zeta |f_u(X_t)| dt < \infty$ and

$$u(x) = E_x \int_0^\zeta f_u(X_t) dt + E_x \int_0^\zeta dA_t^\mu \quad (4.8)$$

for q.e. $x \in E$.

We now introduce an important notion of quasi- L^1 functions on E (we recall that it appears in condition (A3')).

Definition. We say that a Borel function f on E is quasi- L^1 with respect to the regular Dirichlet form $(\mathcal{E}, D[\mathcal{E}])$ on $L^2(E; m)$ if the function $t \mapsto f(X_t)$ belongs to $L^1_{loc}(\mathbb{R}_+)$ P_x -a.s. for q.e. $x \in E$.

Remark 4.3. If $f \in L^1(E; m)$ then f is locally quasi- L^1 . Indeed, if $f \in L^1(E; m)$ then by [10, Theorem 5.1.3],

$$E_m \int_0^T |f(X_t)| dt = \int_0^T \langle |f|, p_t 1 \rangle dt \leq T \|f\|_{L^1(E; m)},$$

where $\{p_t, t \geq 0\}$ is the semigroup associated with the operator A corresponding to \mathcal{E} . Therefore $E_x \int_0^T |f(X_t)| dt < \infty$ for m -a.e. $x \in E$, and hence, by Lemma 4.2, for q.e. $x \in E$.

Remark 4.4. A different notion of quasi-integrability was introduced in the paper [19] devoted to semilinear elliptic systems with measure data. In [19], where the Laplace operator Δ on a smooth bounded domain $D \subset \mathbb{R}^d$ subject to the Dirichlet boundary conditions is considered, a measurable function $f : D \rightarrow \mathbb{R}$ is called quasi- L^1 if for every $\varepsilon > 0$ and compact set $K \subset D$ there exists an open set $U \subset D$ such that $\text{cap}(U) < \varepsilon$ and $f \in L^1(K \setminus U; dx)$. By [19, Proposition 2.3], f is quasi- L^1 on D iff there exists a quasi-finite function G on D and $H \in L^1(D; dx)$ such that $|f| \leq G + H$, m -a.e., where m is the Lebesgue measure on D and cap is the Newtonian capacity, i.e. the capacity associated with the form generating Δ (see Section 6). Here “quasi-finite” means that for every $\varepsilon > 0$ and every compact set $K \subset D$ there exists $M > 0$ and an open set $U \subset D$ such that $\text{cap}(U) < \varepsilon$ and $|G| \leq M$, m -a.e. on $K \setminus U$.

Let us observe that if f is quasi- L^1 in the sense of [19] than for every compact subset $K \subset D$, the function $f|_K$ is quasi- L^1 in the sense defined in our paper. To see this, let us first note that by Remark 4.3, H is quasi- L^1 in the sense of our definition. Since G is quasi-finite, there exists a decreasing sequence $\{U_n\}$ of open subsets of D and a sequence $\{M_n\}$ of positive constants such that $\text{cap}(U_n) \searrow 0$ and $G|_{K \setminus U_n} \leq M_n$, m -a.e.. In particular, $G \in L^1(K \setminus U_n; dx)$ for $n \in \mathbb{N}$. From this and [10, Theorem 4.2.1] it follows that for q.e. $x \in E$,

$$\begin{aligned} P_x\left(\int_0^T G|_K(X_t) dt = \infty\right) &\leq P_x\left(\int_0^T G|_{K \setminus U_n}(X_t) dt = \infty\right) + P_x\left(\int_0^T G|_{U_n}(X_t) dt = \infty\right) \\ &= P_x\left(\int_0^T G|_{U_n}(X_t) dt = \infty\right) \\ &\leq P_x(\exists t \in [0, T] X_t \in U_n) = P_x(\sigma_{U_n} \leq T) \leq e^T E_x e^{-\sigma_{U_n}}, \end{aligned}$$

which converges to zero as $n \rightarrow \infty$. Thus, $G|_K$ is quasi- L^1 , which completes the proof that $f|_K$ is quasi- L^1 .

The class of quasi- L^1 functions defined in [19] is well adjusted to the Dirichlet problem with zero boundary conditions. It is, however, too large to get existence results in our general setting (for instance if the Dirichlet form leads to equations with the Laplace operator subject to Neumann boundary conditions). To overcome this difficulty one can define analytically a bit narrower class of functions, say the class qL^1 , consisting of all measurable $f : D \rightarrow \mathbb{R}$ such that for every $\varepsilon > 0$ there exists an open set $U \subset D$ such that $\text{cap}(U) < \varepsilon$ and $f \in L^1(D \setminus U; dx)$. Then in the same manner as above (with $K = D$) one can show that if $f \in qL^1$ then f is quasi- L^1 in the sense of our definition. The class qL^1 is in general narrower than the class of quasi- L^1 defined in the present paper. To see this it suffices to consider the Dirichlet form (6.2) with $D = \mathbb{R}^d$ and coefficients a_{ij} satisfying (6.3) and condition (b). Then every continuous function f on \mathbb{R}^d is quasi- L^1 but $f \equiv 1$ does not belong to qL^1 .

Remark 4.5. The space of quasi- L^1 functions is quite wide. It contains many singular functions. In case A is as in Remark 4.4, a typical example of such function is $f : B^d(0, 1) \rightarrow \mathbb{R}$, $d \geq 2$, defined by $f(x) = |x|^{-\alpha}$ for some $\alpha > 0$.

In order to state succinctly our main theorem on existence and uniqueness of solutions of (4.7), we introduce the following terminology. We say that a function $u : E \rightarrow \mathbb{R}$

is of class (FD) if the process $t \mapsto u(X_t)$ is of class (D) under the measure P_x for q.e. $x \in E$. Similarly, we say that $u \in \mathcal{FD}^p$ if the process $t \mapsto u(X_t)$ belongs to \mathcal{D}^p under P_x for q.e. $x \in E$.

Theorem 4.6. *Assume (A1), (A2), (A3'), (A4'). Then there exists a unique solution u of (4.7) such that u is of class (FD). Actually, $u \in \mathcal{FD}^q$ for $q \in (0, 1)$. Moreover, for q.e. $x \in E$ there exists a unique solution (Y^x, M^x) of BSDE($\zeta, f^0 + dA^\mu$) on $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$. In fact,*

$$u(X_t) = Y_t^x, \quad t \geq 0, \quad P_x\text{-a.s.}$$

Proof. By Lemma 4.1, condition (A4') is satisfied q.e.. Let us denote by N the set of those $x \in E$ for which (A4') is not satisfied. In view of [10, Theorem 4.1.1] we may assume that N is properly exceptional. By Theorem 3.4, for $x \in N^c$ there exists a unique solution (Y^x, M^x) of BSDE($\zeta, f^0 + dA^\mu$) on $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$ such that $Y^x \in \mathcal{D}^q, q \in (0, 1)$, Y^x is of class (D) and $M^x \in \mathcal{M}^q, q \in (0, 1)$. By Remark 3.6 there exists a pair (Y, M) of (\mathcal{F}_t) adapted càdlàg processes which is a version of (Y^x, M^x) under P_x for every $x \in N^c$. Let us put

$$u(x) = E_x Y_0, \quad x \in N^c, \quad u(x) = 0, \quad x \in N.$$

By the Markov property, Proposition 3.5 and the fact that N is properly exceptional, for every $x \in N^c$ we have

$$u(X_t) = E_{X_t} Y_0 = E_{X_t} Y_t^t = E_x(Y_t^t \circ \theta_t | \mathcal{F}_t) = E_x(Y_t^0 | \mathcal{F}_t) = Y_t^0, \quad P_x\text{-a.s.}$$

Since $u \in \mathcal{C}$, $u(X_t) = Y_t^x, t \geq 0, P_x\text{-a.s.}$ for q.e. $x \in E$, and the proof is complete. \square

Let us note that (A3'), (A4') are minimal assumptions under which there exists an m -a.e. finite solution of (4.7). In the next section we formulate some purely analytic conditions on f, μ which for transient Dirichlet forms imply (A3'), (A4').

Remark 4.7. (i) A remarkable feature of Theorem 4.6 is that it can be used in situations in which the underlying Dirichlet form is not transient. For instance, it applies to Dirichlet problem with Laplace operator in dimensions 1 and 2.

(ii) Suppose that f does not depend on y and $\mu \equiv 0$. One of the equivalent conditions ensuring transiency of $(\mathcal{E}, D[\mathcal{E}])$ is that for every nonnegative $f \in L^1(E; m)$ condition (A4') is satisfied. This shows that if $d = 1$ or $d = 2$ then one can find $f \in L^1(E; m)$ such that there is no solution of (4.7).

Proposition 4.8. *Let u_1, u_2 be solutions of (4.7) with the data $(f^1, \mu_1), (f^2, \mu_2)$, respectively, such that u_1, u_2 are of class (FD). Assume that $\mu_1 \leq \mu_2$ and either $f^1(x, u_1(x)) \leq f^2(x, u_1(x))$ m -a.e. and f^2 satisfies condition (c) of Theorem 4.6 or $f^1(x, u_2(x)) \leq f^2(x, u_2(x))$ m -a.e. and f^1 satisfies (c). Then $u_1(x) \leq u_2(x)$ for q.e. $x \in E$.*

Proof. By Theorem 4.6, $u_1(X), u_2(X)$ are first components of the solutions of BSDE($\zeta, f^{1,0} + dA^{\mu_1}$) and BSDE($\zeta, f^{2,0} + dA^{\mu_2}$), respectively. Since $(f_{u_1}^1 - f_{u_2}^2)^+ = 0, m$ -a.e., then by uniqueness of the Revuz duality, for q.e. $x \in E, \int_0^t (f_{u_1}^1 - f_{u_2}^2)^+(X_s) ds = 0, t \geq 0, P_x\text{-a.s.}$ It follows that for q.e. $x \in E$ the solutions of the backward equations satisfy on the space $(\Omega, \mathcal{F}, \mathcal{F}_t, P_x)$ the assumptions of Proposition 3.1. Therefore $u_1(X_t) \leq u_2(X_t), t \geq 0, P_x\text{-a.s.}$ for q.e. $x \in E$, and consequently, $u_1(x) \leq u_2(x)$ for q.e. $x \in E$. \square

5 Regularity of probabilistic solutions

In this section we investigate regularity properties of probabilistic solutions of (4.7) under the additional assumption that $(\mathcal{E}, D[\mathcal{E}])$ is transient and μ is a bounded smooth measure. We also show that under these assumptions the probabilistic solution of (4.7) can be defined purely analytically by duality.

We begin with definitions of some subsets of the set S of smooth measures. For more details we refer the reader to [10].

$\mathcal{M}_{0,b}$ denote the class of all smooth measures on E such that $|\mu|(E) < \infty$, where $|\mu|$ stands for the total variation of μ (elements of $\mathcal{M}_{0,b}$ are sometimes called soft measures; see [9]). $\mathcal{M}_{0,b}^+$ denote the subset of $\mathcal{M}_{0,b}$ consisting of all nonnegative measures.

Let us recall that a Markovian semigroup $\{p_t, t \geq 0\}$ is called transient if for every nonnegative $f \in L^1(E; m)$,

$$Gf(x) \equiv \lim_{N \rightarrow +\infty} S_N f(x) < \infty, \quad m\text{-a.e.}$$

(the limit above is well defined since the sequence is monotone), where

$$S_t f = \int_0^t p_s f \, ds, \quad t \geq 0.$$

We say that a Dirichlet form $(\mathcal{E}, D[\mathcal{E}])$ is transient if its associated semigroup is transient. Assume that $(\mathcal{E}, D[\mathcal{E}])$ is a transient regular form. Then \mathcal{E} can be extended to a function space $\mathcal{F}_e \subset \mathcal{B}(E; m)$ in such a way that $(\mathcal{E}, \mathcal{F}_e)$ is a Hilbert space. It is known that $\mathcal{F}_e \cap L^2(E; m) = D[\mathcal{E}]$ and there exists a strictly positive bounded function $g \in L^1(E; m)$ such that $\mathcal{F}_e \subset L^1(E; g \cdot dm)$ and

$$(|u|, g)_{L^2(E; m)} \leq c \sqrt{\mathcal{E}(u, u)}, \quad u \in \mathcal{F}_e. \quad (5.1)$$

In fact this is an equivalent condition for transiency of the form $(\mathcal{E}, D[\mathcal{E}])$ (see [10, Theorem 1.5.1]). By $S_0^{(0)}$ we denote the set of all nonnegative smooth measures such that

$$\int_E |v(x)| \mu(dx) \leq c \sqrt{\mathcal{E}(v, v)}, \quad v \in \mathcal{F}_e \cap C_0(E) \quad (5.2)$$

for some $c > 0$. By Riesz's theorem, for every $\mu \in S_0^{(0)}$ there exists a unique function $U\mu \in \mathcal{F}_e$, called the (0-order) potential of the measure μ , such that

$$\mathcal{E}(U\mu, v) = \int_E v(x) \mu(dx), \quad v \in \mathcal{F}_e \cap C_0(E).$$

By S_0 we denote the class of nonnegative smooth measures such that

$$\int_E |v(x)| \mu(dx) \leq c \sqrt{\mathcal{E}_1(v, v)}, \quad v \in \mathcal{F} \cap C_0(E).$$

Again by Riesz's theorem, for every $\mu \in S_0$ and $\alpha > 0$ there exists a unique function $U_\alpha \mu \in \mathcal{F}$, called α -potential of μ , such that

$$\mathcal{E}_\alpha(U_\alpha \mu, v) = \int_E v(x) \mu(dx), \quad v \in \mathcal{F} \cap C_0(E).$$

Of course $S_0^{(0)} \subset S_0$. By $S_{00}^{(0)}$ we denote the subset of $S_0^{(0)}$ consisting of all measures μ such that $U\mu$ is bounded q.e..

Lemma 5.1. *Let $\mu \in S$, $\nu \in S_{00}^{(0)}$. Then for any nonnegative Borel function f ,*

$$E_\nu \int_0^\zeta f(X_t) dA_t^\mu = \langle f \cdot \mu, U\nu \rangle.$$

Proof. Let $\{F_n\}$ be a generalized nest such that $\mathbf{1}_{F_n}|f| \cdot |\mu|, \mathbf{1}_{F_n}|f|, \mathbf{1}_{F_n} \cdot |\mu| \in S_{00}^{(0)}$. By [10, Lemma 5.1.3], for every $\alpha > 0$,

$$E_x \int_0^\zeta e^{-\alpha t} \mathbf{1}_{F_n} f(X_t) dA_t^\mu = U_\alpha(\mathbf{1}_{F_n} f \cdot \mu)(x)$$

for q.e. $x \in E$. Hence

$$E_\nu \int_0^\zeta e^{-\alpha t} \mathbf{1}_{F_n} f(X_t) dA_t^\mu = \langle U_\alpha(\mathbf{1}_{F_n} \cdot \mu), \nu \rangle = \langle \mathbf{1}_{F_n} \cdot \mu, U_\alpha \nu \rangle.$$

Since $\mathbf{1}_{F_n} \cdot \mu \in S_0^{(0)}$, applying [10, Lemma 2.2.11] yields

$$\langle \mathbf{1}_{F_n} \cdot \mu, U_\alpha \nu \rangle \rightarrow \langle \mathbf{1}_{F_n} \cdot \mu, U\nu \rangle.$$

On the other hand, by the properties of the generalized nest and the Lebesgue dominated convergence theorem,

$$E_\nu \int_0^\zeta e^{-\alpha t} \mathbf{1}_{F_n} f(X_t) dA_t^\mu \rightarrow E_\nu \int_0^\zeta f(X_t) dA_t^\mu,$$

and the proof is complete. \square

Let \mathcal{A} denote the space of all quasi-continuous functions $u : E \rightarrow \mathbb{R}$ such that $|\langle \nu, u \rangle| < \infty$ for every $\nu \in S_{00}^{(0)}$. Let us stress that the space \mathcal{A} depends on the form $(\mathcal{E}, D[\mathcal{E}])$. Observe also that $\mathcal{F}_e \subset \mathcal{A}$.

The following definition may be viewed as an analogue of Stampacchia's definition of a solution of linear elliptic equation with measure data (see [27]).

Definition. Assume that $(\mathcal{E}, D[\mathcal{E}])$ is transient and $\mu \in \mathcal{M}_{0,b}$. We say that $u : E \rightarrow \mathbb{R}$ is a solution of (4.7) in the sense of duality if $u \in \mathcal{A}$, $f_u \in L^1(E; m)$ and

$$\langle \nu, u \rangle = (f_u, U\nu)_{L^2(E; m)} + \langle \mu, U\nu \rangle, \quad \nu \in S_{00}^{(0)}. \quad (5.3)$$

Remark 5.2. Solutions in the sense of duality of linear nonlocal elliptic equations with measure data are considered in [14] in case $A = \Delta^\alpha$ on \mathbb{R}^d with $\alpha \in (\frac{1}{2}, 1)$ and $d \geq 2$. It is known (see [10, Exercise 2.2.1]) that in this case

$$Uf(x) = c(d, \alpha) \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{d-2\alpha}} dy, \quad f \in C_0(\mathbb{R}^d).$$

From this one can easily deduce that $C_0^+(\mathbb{R}^d) \subset S_{00}^{(0)}$. It follows in particular that if $u \in \mathcal{A}$ then $u \in L_{loc}^1(E; m)$. It is also known (see [10, Exercise 1.5.2]) that the form $(\mathcal{E}, D[\mathcal{E}])$ corresponding to A is transient. Therefore in case A has the special form considered in [14] our definition of a solution by duality agrees with the one introduced in [14].

Proposition 5.3. *Assume that $(\mathcal{E}, D[\mathcal{E}])$ is transient and $\mu \in \mathcal{M}_{0,b}$. If u is quasi-continuous and $f_u \in L^1(E; m)$, then u is a probabilistic solution of (4.7) iff u is a solution of (4.7) in the sense of duality.*

Proof. Let u be a solution of (4.7) in the sense of duality. Let us denote by $w(x)$ the right-hand side of (4.8) if it is finite and put $w(x) = 0$ otherwise. By Proposition 5.13, w is finite m -a.e., and hence, by Lemma 4.2, w is quasi-continuous. By Lemma 5.1, $w \in \mathcal{A}$ and

$$\langle \nu, w \rangle = (f_u, U\nu)_{L^2(E; m)} + \langle \mu, U\nu \rangle, \quad \nu \in S_{00}^{(0)}.$$

Thus, $\langle \nu, u \rangle = \langle \nu, w \rangle$ for $\nu \in S_{00}^{(0)}$. By [10, Theorem 2.2.3], this implies that $u = w$ q.e. since u, w are quasi-continuous.

Conversely, assume that u is a probabilistic solution of (4.7). Then again by Lemma 5.1, $u \in \mathcal{A}$ and u satisfies (5.3). \square

In view of Proposition 5.3 there arise natural questions. When $f_u \in L^1(E; m)$? Is the assumption $\mu \in \mathcal{M}_{0,b}$, $f(\cdot, 0) \in L^1(E; m)$ sufficient for integrability of f_u ? It is always true that a probabilistic solution u of (4.7) or a solution in the sense of duality is locally integrable? We will show that if $\mu \in \mathcal{M}_{0,b}$, $f(\cdot, 0) \in L^1(E; m)$ then $f_u \in L^1(E; m)$ but u need not be locally integrable.

Let μ be a Borel measure on E . In the sequel, $\|\mu\|_{TV}$ stands for its total variation norm.

Lemma 5.4. *Assume that $(\mathcal{E}, D[\mathcal{E}])$ is transient, $\mu_1 \in S$, $\mu_2 \in \mathcal{M}_{0,b}^+$. If*

$$E_x \int_0^\zeta dA_t^{\mu_1} \leq E_x \int_0^\zeta dA_t^{\mu_2}$$

for m -a.e. $x \in E$ then $\|\mu_1\|_{TV} \leq \|\mu_2\|_{TV}$.

Proof. By Lemma 4.1 and [10, Lemma 2.1.4], $E_x \int_0^\zeta dA_t^{\mu_1} \leq E_x \int_0^\zeta dA_t^{\mu_2}$ for q.e. $x \in E$ and hence, by Lemma 5.1,

$$\langle \mu_1, U\nu \rangle \leq \langle \mu_2, U\nu \rangle \tag{5.4}$$

for every $\nu \in S_{00}^{(0)}$. Since E is locally compact and $(\mathcal{E}, D[\mathcal{E}])$ is regular, there is a sequence $\{U_k\}$ of decreasing open sets such that $\text{cap}(U_k) < \infty$ and $\bigcup_{k \geq 1} U_k = E$. Let $e_k^{(0)}$ be the (0-order) equilibrium associated with the set U_k (see [10] page 71). Then by the 0-order counterpart of [10, Lemma 2.1.1] (see comments before Lemma 2.1.8 in [10]), $0 \leq e_k^{(0)} \leq 1$ q.e., $e^{(0)}(x) = 1$ for q.e. $x \in U_k$, and $e_k^{(0)} = U(\beta_k)$, where $\beta_k \in S_{00}^{(0)}$ is the measure associated with the 0-order potential $e_k^{(0)}$. By (5.4),

$$\langle \mu_1, U(\beta_k) \rangle \leq \langle \mu_2, U(\beta_k) \rangle, \quad k \geq 1.$$

Letting $k \rightarrow \infty$ and using Fatou's lemma gives the desired result. \square

Proposition 5.5. *Assume that $(\mathcal{E}, D[\mathcal{E}])$ is transient, $\mu \in \mathcal{M}_{0,b}$ and $f(\cdot, 0) \in L^1(E; m)$. If u is a probabilistic solution of (4.7), then $f_u \in L^1(E; m)$ and*

$$\|f_u\|_{L^1(E; m)} \leq \|f(\cdot, 0)\|_{L^1(E; m)} + \|\mu\|_{TV}.$$

Proof. By Lemma 2.3, Proposition 5.13 and Theorem 4.6,

$$E_x \int_0^\zeta |f_u(X_t)| dt \leq E_x \int_0^\zeta |f(X_t, 0)| dt + E_x \int_0^\zeta d|A^\mu|_t$$

for m -a.e. $x \in E$. Therefore the desired inequality follows from Lemma 5.4. \square

Corollary 5.6. *If $(\mathcal{E}, D[\mathcal{E}])$ is transient and (A4) is satisfied, then u is a probabilistic solution of (4.7) iff it is a solution of (4.7) in the sense of duality.*

Example 5.7. To show that in general a probabilistic solution of (4.7) is not locally integrable let us consider the following trivial form

$$\mathcal{E}(u, v) = \int_{-1}^1 c(x)u(x)v(x) dx, \quad u, v \in D[\mathcal{E}] = L^2(D; m),$$

where $D = (-1, 1)$, $c(x) = |x|$ and m is the Lebesgue measure. Then $(\mathcal{E}, D[\mathcal{E}])$ is a transient regular Dirichlet form and by Theorem 4.6 there exists a unique solution u of the equation

$$-Au = 1.$$

Obviously, u is given by the formula

$$u(x) = |x|^{-1}, \quad x \in D,$$

and so is not locally integrable.

Remark 5.8. Local integrability of u is related to the condition

$$\forall K \subset E, K\text{-compact}, \quad R\mathbf{1}_K \in L^\infty(E; m). \quad (5.5)$$

To see this, let us consider a transient regular Dirichlet form $(\mathcal{E}, D[\mathcal{E}])$. Suppose that for any $f \in L^1(E; m)$ a solution u of the problem

$$-Au = f$$

is locally integrable. Then by [10, Theorem 5.1.3], for every compact $K \subset E$ and nonnegative $f \in L^1(E; m)$,

$$\int_K |u| dm = \int_K u dm = (f, R\mathbf{1}_K)_{L^2(E; m)} < \infty,$$

which implies that (5.5) is satisfied. Conversely, assume that (5.5) is satisfied. Let u be a solution of the problem (4.7) with f, μ satisfying the assumptions of Proposition 5.5. Then applying [10, Theorem 5.1.3] shows that for every compact $K \subset E$,

$$\int_K |u| dm \leq (|f_u|, R\mathbf{1}_K)_{L^2(E; m)} + \langle |\mu|, R\mathbf{1}_K \rangle,$$

and hence (5.5) is satisfied since $f_u \in L^1(E; m)$. Some examples of forms satisfying (5.5) will be given in Section 6.

Before going to the next result let us note that by the very definition of the extended Dirichlet space, for every $u \in \mathcal{F}_e$ there exists an approximating sequence $\{u_n\} \subset D[\mathcal{E}]$ such that $u_n \rightarrow u$ in \mathcal{E} and m -a.e.. If u is bounded m -a.e. by some constant $k > 0$ then by Theorem 1.4.2 and Corollary 1.5.1 in [10], $\{T_k(u_n)\} \subset D[\mathcal{E}]$, $T_k(u_n) \rightarrow u$ in \mathcal{E} , and hence m -a.e.. Thus, for every bounded $u \in \mathcal{F}_e$ we can always find an approximating sequence which is bounded m -a.e. by the same constant as u .

Proposition 5.9. *Assume that $(\mathcal{E}, D[\mathcal{E}])$ is transient and $\mu \in \mathcal{M}_{0,b}$. Then if u is a solution of (4.7) and $f_u \in L^1(E; m)$ then $T_k(u) \in \mathcal{F}_e$ for every $k \geq 0$. Moreover, for every $k \geq 0$,*

$$\mathcal{E}(T_k(u), T_k(u)) \leq k(\|f_u\|_{L^1(E; m)} + \|\mu\|_{TV}). \quad (5.6)$$

Proof. Let $\{F_n\}$ be a generalized nest such that $\mathbf{1}_{F_n}|f_u| \cdot m + \mathbf{1}_{F_n}|\mu| \in S_{00}^{(0)}$. Set

$$u_n(x) = E_x \int_0^\zeta \mathbf{1}_{F_n} f_u(X_t) dt + E_x \int_0^\zeta \mathbf{1}_{F_n}(X_t) dA_t^\mu, \quad x \in E$$

and define v_n, w_n as u_n but with f_u, μ replaced by f_u^+, μ^+ and f_u^-, μ^- , respectively. Of course, $u_n = v_n - w_n$. By Lemma 5.1 and [10, Theorem 2.2.3],

$$v_n(x) = U(\mathbf{1}_{F_n} f_u^+ \cdot m + \mu^+)(x), \quad w_n(x) = U(\mathbf{1}_{F_n} f_u^- \cdot m + \mu^-)(x)$$

for q.e. $x \in E$. Therefore $u_n \in \mathcal{F}_e$. By remarks following Remark 5.2 we can find a sequence $\{\varepsilon_m(T_k(u_n))\}_m \subset D[\mathcal{E}]$ approximating $T_k(u_n)$ in \mathcal{E} such that $|\varepsilon_m(T_k(u_n))| \leq k$, m -a.e. for $n, m \in \mathbb{N}$. Since $R_\alpha(L^2(E; m)) \subset D(A)$ and αR_α is Markovian, for every $\alpha > 0$, $\alpha R_\alpha[\varepsilon_m(T_k(u_n))] \in D(A)$ and $|\alpha R_\alpha[\varepsilon_m(T_k(u_n))]| \leq k$, m -a.e.. It follows that the measure $\nu = A(\alpha R_\alpha[\varepsilon_m(T_k(u_n))])$ belongs to $S_{00}^{(0)} - S_{00}^{(0)}$. Taking ν as a test measure in (5.3) we get

$$\mathcal{E}(u_n, \alpha R_\alpha[\varepsilon_m(T_k(u_n))]) \leq k(\|f_u\|_{L^2(E; m)} + \|\mu\|_{TV}), \quad k \geq 0. \quad (5.7)$$

By the strong continuity of the resolvent (see [10, Lemma 1.3.3]),

$$\lim_{\alpha \rightarrow \infty} \mathcal{E}(u_n, \alpha R_\alpha[\varepsilon_m(T_k(u_n))]) = \mathcal{E}(u_n, \varepsilon_m(T_k(u_n))),$$

and by the definition of the sequence $\{\varepsilon_m(T_k(u_n))\}$,

$$\lim_{m \rightarrow \infty} \mathcal{E}(u_n, \varepsilon_m(T_k(u_n))) = \mathcal{E}(u_n, T_k(u_n)).$$

Moreover, from the Beurling-Deny representation of the form \mathcal{E} (see [10, Theorem 3.2.1]) it follows that

$$\mathcal{E}(T_k(u_n), T_k(u_n)) \leq \mathcal{E}(u_n, T_k(u_n)).$$

Hence

$$\sup_{n \geq 1} \mathcal{E}(T_k(u_n), T_k(u_n)) < \infty$$

for every $k > 0$. On the other hand, as in proof of (4.2) one can show that $u_n(x) \rightarrow u(x)$ q.e.. Therefore (5.6) follows from (5.1) and the fact that $(\mathcal{E}, \mathcal{F}_e)$ is a Hilbert space. \square

Proposition 5.10. *Under the assumptions of Proposition 5.9 the following condition of vanishing energy is satisfied:*

$$\mathcal{E}(\Phi_k(u), \Phi_k(u)) \leq \int_{\{|u| \geq k\}} |f_u(x)| m(dx) + \int_{\{|u| \geq k\}} d\mu, \quad (5.8)$$

where $\Phi_k(r) = T_1(r - T_k(r))$, $r \in \mathbb{R}$.

Proof. Let us define u_n as in the proof of Proposition 5.9 and let $\{\varepsilon_m(\Phi_k(u_n))\}_m \subset D[\mathcal{E}]$ be an approximating sequence for $\Phi(u_n)$. Since $\Phi(u_n)$ is bounded, we may assume that the sequence $\{\varepsilon_m(\Phi_k(u_n))\}_m$ is bounded. Then $\alpha R_\alpha(\varepsilon_m(\Phi_k(u_n))) \in D(A)$ and $\alpha R_\alpha(\varepsilon_m(\Phi_k(u_n)))$ is bounded by the same constant as $\varepsilon_m(\Phi_k(u_n))$. Therefore $\nu \equiv A[\alpha R_\alpha(\varepsilon_m(\Phi_k(u_n)))] \in S_{00}^{(0)} - S_{00}^{(0)}$. By Proposition 5.3,

$$\langle u_n, \nu \rangle = (f_u \mathbf{1}_{F_n}, \alpha R_\alpha(\varepsilon_m(\Phi_k(u_n))))_{L^2(E; m)} + \langle \mu \mathbf{1}_{F_n}, \alpha R_\alpha(\varepsilon_m(\Phi_k(u_n))) \rangle.$$

It is known (see [10, Lemma 1.3.3]) that $\alpha R_\alpha(\varepsilon_m(\Phi_k(u_n))) \rightarrow \varepsilon_m(\Phi_k(u_n))$ in \mathcal{E}_1 and q.e.. By the definition, $\varepsilon_m(\Phi_k(u_n)) \rightarrow \Phi_k(u_n)$ in \mathcal{E} , which by [10, Theorem 2.1.4] (see also comments following [10, Corollary 2.2.2]) implies that the convergence holds q.e. as well. Since the considered sequences are bounded and $f_u \in L^1(E; m)$, $\mu \in \mathcal{M}_{0,b}$, applying the Lebesgue dominated convergence theorem we obtain

$$\mathcal{E}(u_n, \Phi_k(u_n)) = (f_u \mathbf{1}_{F_n}, \Phi_k(u_n))_{L^2(E; m)} + \langle \mu \mathbf{1}_{F_n}, \Phi_k(u_n) \rangle.$$

Since $u_n \rightarrow u$ q.e. (see the proof of (4.2)), we get

$$\begin{aligned} (f_u \mathbf{1}_{F_n}, \Phi_k(u_n))_{L^2(E; m)} + \langle \mu \mathbf{1}_{F_n}, \Phi_k(u_n) \rangle &\leq \int_{\{|u_n| \geq k\}} |f_u(x)| m(dx) + \int_{\{|u_n| \geq k\}} d\mu \\ &\rightarrow \int_{\{|u| \geq k\}} |f_u(x)| m(dx) + \int_{\{|u| \geq k\}} d\mu. \end{aligned}$$

On the other hand, by the Buerling-Deny representation of the form \mathcal{E} and Proposition 5.9,

$$\mathcal{E}(u_n, \Phi_k(u_n)) \geq \mathcal{E}(\Phi_k(u_n), \Phi_k(u_n)) \rightarrow \mathcal{E}(\Phi_k(u), \Phi_k(u)),$$

which completes the proof of (5.8). \square

Remark 5.11. From Proposition 5.10 it follows in particular that if A is a uniformly elliptic divergence form operator on $D \subset \mathbb{R}^d$ with $d \geq 3$ (i.e. A corresponds to the form $(\mathcal{E}(D), D[\mathcal{E}])$ defined by (6.2) with coefficients a_{ij} satisfying (6.3)), then the probabilistic solution of (4.7) is a renormalized solution (see [2]) of (4.7), because in that case $D[\mathcal{E}] = \mathcal{F}_e = H_0^1(D)$ by Poincaré's inequality. It is worth pointing out that Propositions 5.9 and 5.10 suggest possibility of extending the definition of renormalized solutions to general operators corresponding to transient regular Dirichlet forms, notably to some nonlocal operators. Let us also note that renormalized solutions of some elliptic equations with L^1 -data and A being a fractional Laplacian on \mathbb{R}^d are studied in [1].

Remark 5.12. Let u be a solution of (4.7). If $f_u \in L^2(E; m)$ and $\mu \in S_0^{(0)}$ then by Lemma 5.1 and [10, Theorem 2.2.5], $U(f + \mu) \in \mathcal{F}_e$, $u = U(f + \mu)$ q.e. and for every $v \in \mathcal{F}_e$,

$$\mathcal{E}(u, v) = (f_u, v)_{L^2(E; m)} + \langle v, \mu \rangle,$$

i.e. u is the usual weak solution of (4.7).

From Remark 4.3 it follows that condition (A3) implies (A3'). That (A4) implies (A4') follows from the proposition given below.

Proposition 5.13. *If $(\mathcal{E}, D[\mathcal{E}])$ is transient and $\mu \in \mathcal{M}_{0,b}^+$ then for m -a.e. $x \in E$,*

$$E_x \int_0^\zeta dA_t^\mu < \infty.$$

Proof. For $x \in E$ set

$$S_t \mu(x) = E_x \int_0^t dA_s^\mu, \quad t \geq 0, \quad G\mu(x) = \lim_{n \rightarrow \infty} S_n \mu(x).$$

We have to prove that $G\mu(x) < \infty$ for m -a.e. $x \in E$. By [10, Theorem 5.1.3] and the fact that the semigroup $\{p_t, t \geq 0\}$ associated with the form \mathcal{E} is Markovian,

$$\|S_t \mu\|_{L^1(E;m)} = E_m \int_0^t dA_s^\mu = \int_0^t \langle \mu, p_s 1 \rangle ds \leq \int_0^t \langle \mu, 1 \rangle = t \|\mu\|_{TV}.$$

We can now repeat the proof of [10, Lemma 1.5.1] with $f \in L^1(E;m)$ replaced by μ and $S_t f$ replaced by $S_t \mu$ to show that if there exists a strictly positive function $g \in L^1(E;m)$ such that $Gg(x) < \infty$, m -a.e., then $G\mu(x) < \infty$, m -a.e. for every $\mu \in S \cap \mathcal{M}_{0,b}^+$. But such function g exists since $(\mathcal{E}, D[\mathcal{E}])$ is transient. \square

Theorem 5.14. *Assume that $(\mathcal{E}, D[\mathcal{E}])$ is transient and μ, f satisfy (A1)–(A4). Then there exists a unique probabilistic solution u of (4.7) such that u is of class (FD) and $u \in \mathcal{FD}^q$, $q \in (0, 1)$. Moreover, $f_u \in L^1(E;m)$ and $T_k(u) \in \mathcal{F}_e$ for every $k \geq 0$.*

Proof. Follows from Proposition 5.13 and Proposition 5.9. \square

In view of Corollary 5.6, the solution u of Theorem 5.14 is a solution of (4.7) in the sense of duality.

Let $(\mathcal{E}, D[\mathcal{E}])$ be a regular Dirichlet form and let g be a strictly positive bounded Borel function on E . Then by [10, Lemma 1.6.6] the perturbed form $(\mathcal{E}^g, D[\mathcal{E}])$, where

$$\mathcal{E}^g(u, v) = \mathcal{E}(u, v) + (u, v)_{L^2(E; g \cdot dm)}$$

is a transient regular Dirichlet form on $L^2(E;m)$. The operator A^g associated with $(\mathcal{E}^g, D[\mathcal{E}])$ has the form $A^g = A + g$, where A is associated with $(\mathcal{E}, D[\mathcal{E}])$. Therefore an immediate consequence of Theorem 5.14 is the following proposition.

Proposition 5.15. *If μ, f satisfy (A1)–(A4) and g is a strictly positive bounded Borel function on E then there exists a unique probabilistic solution of the problem*

$$-Au + gu = f_u + \mu.$$

6 Applications

In this section we give typical examples of regular Dirichlet forms and indicate some situations in which our general results are applicable. We keep the same assumptions on E, m as in Section 5.

Let $\{\nu_t, t > 0\}$ be a symmetric convolution semigroup on \mathbb{R}^d and let ψ denote its Lévy-Khintchine symbol, i.e. for $x \in \mathbb{R}^d$ we have

$$\hat{\nu}_t(x) = \int_{\mathbb{R}^d} e^{i(x,y)} \nu_t(dy) = e^{-t\psi(x)}.$$

It is known (see [10, Example 1.4.1]) that the form

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \hat{u}(x) \hat{v}(x) \psi(x) dx, \quad u, v \in D[\mathcal{E}],$$

$$D[\mathcal{E}] = \{u \in L^2(\mathbb{R}^d; dx); \int_{\mathbb{R}^d} |\hat{u}(x)|^2 \psi(x) dx < \infty\}$$

determined by $\{\nu_t, t > 0\}$ is a regular Dirichlet form on $L^2(\mathbb{R}^d; dx)$. Let us denote by $-\psi(\nabla)$ the nonpositive self-adjoint operator associated with $(\mathcal{E}, D[\mathcal{E}])$.

Proposition 6.1. *Assume that μ, f satisfy (A1)–(A4). If $1/\psi$ is locally integrable on \mathbb{R}^d (or, equivalently, $\int_0^\infty \nu_t(K) dt < \infty$ for any compact set $K \subset \mathbb{R}^d$), then there exists a unique probabilistic solution of the problem*

$$-\psi(\nabla)u = f(x, u) + \mu, \quad x \in \mathbb{R}^d.$$

Proof. In [10, Exercise 1.5.2] it is shown that $(\mathcal{E}, D[\mathcal{E}])$ is transient iff $1/\psi$ is locally integrable on \mathbb{R}^d and that the last condition holds iff $\int_0^\infty \nu_t(K) dt < \infty$ for any compact set $K \subset \mathbb{R}^d$. Therefore the proposition follows from Theorem 5.14. \square

Example 6.2. (i) (fractional Laplacian) Let $\psi(x) = c|x|^\alpha$ for some $\alpha \in (0, 2]$, $c > 0$. The form is transient iff $\alpha < d$. Let us also note that $\psi(\nabla) = c(\nabla^2)^{\alpha/2} = c\Delta^{\alpha/2}$.

(ii) (relativistic Schrödinger operator, see [7]) Let $\psi(x) = \sqrt{m^2 c^4 + c^2 |x|^2} - mc^2$. It is an elementary check that the form determined by ψ is transient if $d \geq 3$.

(iii) (operator associated with the relativistic α -stable process). Let $0 < \alpha < 2$ and let $\psi(x) = (|x|^2 + m^{\alpha/2})^{2/\alpha} - m$. Then the associated form is transient iff $d > 2$ (see [4, Chapter 5]).

(iv) (operator associated with the variance gamma process). Let $\psi(x) = \log(1 + |x|^2)$. Then associated form is transient iff $d > 2$. This type of processes was applied in finance (see [17]).

(v) (operator associated with Brownian motion with Bessel subordinator). Let $\psi(x) = \log((1 + |x|^2) + \sqrt{(1 + |x|^2)^2 - 1})$. Then the associated form is transient iff $d > 1$ (see [4, Chapter 5]).

Let $(\mathcal{E}, D[\mathcal{E}])$ be the form of Proposition 6.1 and let D be an open subset of \mathbb{R}^d . Set $L_D^2(\mathbb{R}^d; dx) = \{u \in L^2(\mathbb{R}^d; dx) : u = 0 \text{ a.e. on } D^c\}$, $D[\mathcal{E}_D] = \{u \in D[\mathcal{E}] : \tilde{u} =$

0 -q.e. on D^c , where \tilde{u} is a quasi-continuous version of u . By [10, Theorem 4.4.3], the form $(\mathcal{E}, D[\mathcal{E}_D])$ is a regular Dirichlet form on $L_D^2(\mathbb{R}^d; dx)$, and by [10, Theorem 4.4.4], if $(\mathcal{E}, D[\mathcal{E}])$ is transient then $(\mathcal{E}, D[\mathcal{E}_D])$ is transient, too. Therefore from Theorem 5.14 we get the following proposition.

Proposition 6.3. *Let $D \subset \mathbb{R}^d$ be an open set and μ, f satisfy (A1)–(A4). If $g : D \rightarrow \mathbb{R}$ is a strictly positive bounded Borel function or $1/\psi$ is locally integrable on D and g is a nonnegative bounded Borel function then there exists a unique probabilistic solution of the problem*

$$-\psi(\nabla)u + gu = f(x, u) + \mu, \quad u|_{D^c} = 0. \quad (6.1)$$

Let D be a domain in \mathbb{R}^d . Let us consider the Markovian symmetric form on $D[\mathcal{E}] = C_0^\infty(D)$ defined by

$$\mathcal{E}(u, v) = \sum_{i,j=1}^d \int_D a_{ij}(x) \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx, \quad (6.2)$$

where a_{ij} are locally integrable functions on D such that for every $x \in D$ and $\xi \in \mathbb{R}^d$,

$$\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq 0, \quad a_{ij}(x) = a_{ji}(x), \quad 1 \leq i, j \leq d. \quad (6.3)$$

It is known (see [10, Problem 3.1.1]) that if one of the following conditions

(a) $a_{ij} \in L_{loc}^2(D)$, $\frac{\partial a_{ij}}{\partial x_i} \in L_{loc}^2(D)$, $1 \leq i, j \leq d$,

(b) there exists $\lambda > 0$ such that $\sum_{i,j=1}^d a_{ij}(x) \xi_i \xi_j \geq \lambda |\xi|^2$, $x \in D$, $\xi \in \mathbb{R}^d$

is satisfied, then the form $(\mathcal{E}, D[\mathcal{E}])$ is closable. Therefore its smallest closed extension $(\bar{\mathcal{E}}, D[\bar{\mathcal{E}}])$ is a regular Dirichlet form on $L^2(D; dx)$ (see Theorems 3.1.1 and 3.1.2 in [10]). Let us also note that if $d \geq 3$ and condition (b) is satisfied then from [10, Theorem 1.6.2] and the Gagliardo-Nirenberg-Sobolev inequality it follows that $(\bar{\mathcal{E}}, D[\bar{\mathcal{E}}])$ is transient (for other conditions ensuring transiency see [10, pp. 57–60]). Applying Theorem 5.14 we get existence of a solution of the Dirichlet problem.

Proposition 6.4. *Let D be a domain in \mathbb{R}^d and let a_{ij} , $1 \leq i, j \leq d$, be measurable functions on D satisfying (6.3). Assume that μ, f satisfy (A1)–(A4) on D . If (a) or (b) is satisfied and g is a strictly positive bounded Borel function or (b) is satisfied, $d \geq 3$ and g is nonnegative, then there exists a unique probabilistic solution of the problem*

$$-\sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) + gu = f(x, u) + \mu \text{ on } D, \quad u|_{\partial D} = 0. \quad (6.4)$$

Note that the obstacle problem for equations of the form (6.4) and its connection with BSDEs is investigated in [25].

Theorem 5.14 also applies to the Neumann problem. Let D be a bounded domain in \mathbb{R}^d with boundary of class C , i.e. locally given by a continuous mapping. Let us consider the Markovian symmetric form on $D[\mathcal{E}] = C_0^\infty(\bar{D})$ defined by (6.2) with a_{ij} satisfying (6.3) and condition (b) on \bar{D} . It is known (see [10, Example 1.6.1]) that the form is closable and $(\bar{\mathcal{E}}, D[\bar{\mathcal{E}}]) = (\mathcal{E}, H^1(D))$ is a regular Dirichlet form on $L^2(\bar{D}; dx)$.

Proposition 6.5. *Let $D \subset \mathbb{R}^d$ be a bounded domain of class C and let a_{ij} , $1 \leq i, j \leq d$, be measurable functions on \bar{D} satisfying (6.3) and condition (b). Assume that μ, f satisfy (A1)–(A4) on \bar{D} and g is a strictly positive bounded Borel function on \bar{D} . Then there exists a unique probabilistic solution of the problem*

$$-\sum_{i,j=1}^d \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial u}{\partial x_j}) u + gu = f(x, u) + \mu \text{ on } D, \quad \frac{\partial u}{\partial n} = 0 \text{ on } \partial D.$$

Remark 6.6. (i) Let us consider the operator $\Delta^{\alpha/2}$, $\alpha \in (0, 2)$, on bounded domain $D \subset \mathbb{R}^d$. Then for every compact $K \subset D$,

$$R\mathbf{1}_K(x) = E_x \int_0^\zeta \mathbf{1}_K(X_t) dt \leq E_x \zeta \leq E_x \tau_{B(r)} \leq c(d, \alpha)(r^2 - |x|^2)^{\alpha/2}, \quad x \in B(r)$$

where X is an isotropic α -stable Lévy process on \mathbb{R}^d , $D \subset B(r) = \{x \in \mathbb{R}^d; |x| \leq r\}$ and $\tau_{B(r)} = \inf\{t > 0, X_t \notin B(r)\}$ (see, e.g., [11]). Accordingly, condition (5.5) is satisfied. In fact, the above inequalities show that $R\mathbf{1} \in L^\infty(D; dx)$. Therefore, if f, μ satisfy the assumptions of Proposition 5.5 then $f_u \in L^1(E; dx)$, and consequently,

$$\int_D |u| dm \leq (|f_u|, R\mathbf{1})_{L^2(D; dx)} + \langle |\mu|, R\mathbf{1} \rangle < \infty.$$

Thus, the solution of (6.1) with $\psi(x) = |x|^\alpha$, $\alpha < d$, $g \equiv 0$ belongs to $L^1(D; dx)$. The same conclusion can be drawn for other operators of Example 6.2 considered on bounded domain $D \subset \mathbb{R}^d$ with d specified in the example. As above, to show this it suffices to prove that $x \mapsto E_x \tau_{B(r)}$ is bounded on D . But the last statement follows from results proved in [23].

(ii) Let $D \subset \mathbb{R}^d$, $d \geq 3$, be a bounded domain and let A corresponds to the form (6.2) with coefficients a_{ij} satisfying condition (b). Since it is known that in this case $x \mapsto E_x \tau_D$ is bounded, then under the assumptions of Proposition 6.4 solutions of the problem (6.4) are in $L^1(D; dx)$.

Other interesting situations in which we encounter regular Dirichlet forms include Laplace-Beltrami operators on manifolds (see [10]), quantum graphs (see [15]), perturbations of operators by Radon measures, Hamiltonians with singular interactions (see [5, 28]), diffusion equations with Wentzell boundary condition (see [29]).

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